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WAVES IN A PLASMA IN A MAGNETIC FIELD

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WAVES IN A PLASMA IN A MAGNETIC FIELD

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ABSTRACT

Title of Thesis: Waves in a Plasma in a Magnetic Field

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Two examples of slightly nonlinear wave propagation in a collisionless plasma permeated by a uniform background magnetic field $\underline{B}_0$ are discussed. The first calculation considers externally generated electromagnetic waves propagating along $\underline{B}_0$ in a plasma described by the first three moment equations obtained from the Vlasov Equation. Finite wave amplitude effects would lead to time and/or space growing solutions if conventional perturbation theory were used. These spurious solutions are eliminated by the introduction of amplitude dependent wavenumber and frequency shifts into the calculation, in the hope that such nonlinear effects might be experimentally measurable. The results for a traveling right-circularly polarized (i. e., cyclotron) wave are found to be (for the low frequency region where $\omega_e^2 \gg c^2k^2 \gg \Omega^2 \gg k^2V^2$, $\Omega^2 \gg \omega^2$),

$$\frac{\Delta k}{k} \cong \frac{1}{4}\left(\frac{b}{B_0}\right)^2 \left(\frac{kV}{\Omega}\right)^2$$

for fixed ω, and

$$\frac{\Delta \omega}{\omega} \cong -\frac{1}{2}\left(\frac{b}{B_0}\right)^2 \left(\frac{kV}{\Omega}\right)^2$$

for fixed k. Here b and B_0 are the magnitudes of the wave and

unperturbed magnetic fields respectively and V is the electron thermal velocity. It is found that the sign and ratio of the magnitudes of these dimensionless shifts can be correctly predicted by assuming that the cyclotron dispersion relation is modified by the finite wave amplitude effects.

The second calculation is motivated by the investigation of the line splitting observed in plasma radiation (Type II radio bursts) from the Sun. Here internally excited longitudinal electrostatic oscillations propagating at arbitrary angles with respect to $\underline{B}_0$ are considered. Collisions between these electrostatic plasma waves generate electromagnetic radiation at approximately the plasma frequency for electron-ion plasma wave collisions, and radiation at approximately twice the plasma frequency for electron-electron interactions. In the presence of a magnetic field the direction of propagation determines the frequency of oscillation of the longitudinal waves. Thus, for propagation parallel to $\underline{B}_0$, $\omega = \omega_e$, and for propagation perpendicular to B_0, $\omega \cong \omega_e + \Omega^2/2\omega_e$, where $\Omega = eB_0/mc < \omega_e = (4\pi e^2 n_0/m)^{\frac{1}{2}}$. It is found that suitable anisotropic distributions of energetic or superthermal electrons coexisting with a thermal background can not only drive the fluctuation spectrum of the longitudinal oscillations (and hence the intensity of the emitted radiation) up to very high levels, but can also concentrate the propagation vectors both parallel and perpendicular to $\underline{B}_0$. This concentration produces enhanced emission at ω_e and $\omega_e + \Omega^2/2\omega_e$ and at $2\omega_e$ and $2\omega_e + \Omega^2/\omega_e$

(for the fundamental and second harmonic respectively) with a significant reduction for the intermediate frequencies.

Specific superthermal electron distributions which produce the desired splitting are found to be those with a kinetic temperature perpendicular to $\underline{B}_0$ much larger than that parallel to $\underline{B}_0$, which also have a net drift through the background plasma along $\underline{B}_0$. It is believed that these types of anisotropic distributions might plausibly exist in Type II events, thus explaining their structure.

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CHAPTER I

INTRODUCTION

There are a multitude of wavemotions capable of propagating in a plasma which is permeated by a magnetic field. In this thesis we discuss two calculations, both of which deal with slightly nonlinear wave-propagation in such a magnetized plasma. The first calculation was motivated by an attempt to obtain an experimentally measurable non-linear effect and the second to understand the line splitting observed in plasma radiation (Type II bursts) from the Sun.

The first situation covered will be that of externally generated electromagnetic waves propagating along $\underline{B}_0$ in a collisionless plasma described by the first three moment equations obtained from the Vlasov Equation. Only circularly polarized electromagnetic waves (i.e., cyclotron waves) will propagate along the magnetic field in lowest order. We will find that finite wave amplitude effects will lead to time and/or space growing solutions if we use conventional perturbation theory. Since the system is stable, it appears that this so-called secular behavior is a consequence of the mathematics rather than of the physics. To avoid these spurious unstable solutions we will use a method developed by Krylov, Bogoliubov, and Mitropolsky[1,2] and extended by Montgomery and Tidman.[3] An amplitude dependent frequency or wavenumber shift is introduced into the calculation to enable us to eliminate just those terms which would otherwise cause trouble in the final perturbation solutions. The basic motivation for this section is the hope that such shifts might be experimentally observable.

We shall be primarily concerned with situations in which a standing wave is excited in a bounded slab of plasma, or a traveling wave is excited at some fixed frequency at the boundary of a semi-infinite plasma. Within the framework of the Krylov-Bogoliubov-Mitropolsky perturbation theory[3] for such situations, nonlinearities enter in two ways: (i) Harmonics of the fundamental wavenumber and a complicated spectrum of frequencies are generally present for the standing wave. For the traveling wave excited at a boundary, harmonics of the fundamental frequency and a complicated wavenumber spectrum are generally present. (ii) Frequency or wavenumber shifts in the fundamental phase of the wave are produced. These shifts are amplitude dependent.

Thus suppose for example a standing wave of fixed k is excited in a slab of thickness $2\pi/k$ and that in the linear limit it has a frequency $\omega(k)$ where the dispersion relation $\mathscr{E}(\omega, k) = 0$. Then if the fundamental wave is excited to a finite amplitude a it will oscillate at a shifted frequency $\omega(k) + \Delta\omega(a, k)$. Similarly if one drives a wave at the boundary of a semi-infinite plasma with a fixed frequency ω, the corresponding shifted wavenumber for the plasma wave would be $k(\omega) + \Delta k(a, \omega)$. Montgomery and Tidman[3] showed that the methods of

Ref. 2 could be generalized to include partial differential equations, in particular, the Klein Gorden equation with a small nonlinear source term. The results were applied to a traveling electromagnetic wave in a "cold" plasma (i.e., the pressure tensor is dropped). The shifts obtained were found to be quite small and probably very difficult to measure — even using laser beam intensities. It is hoped that the frequency and wave-number shifts calculated in this work might be more amenable to measurement.

In the low frequency region, $\omega_e^2 >> k^2c^2 >> \Omega^2 >> k^2V^2$, $\Omega^2 >> \omega^2$ (where $\Omega = eB_0/mc$ and $\omega_e^2 = 4\pi n_0 e^2/m$), the right circularly polarized wave has a dispersion relation

$$\omega \cong \frac{c^2k^2}{\omega_e^2}\,\Omega \tag{1}$$

This type of wave is well known in solid state plasma terminology as a "helicon wave."

We have calculated the wavenumber shift for the case of a traveling cyclotron wave excited at some fixed frequency at the boundary of a semi-infinite plasma, and the frequency shift for the case where the wavenumber k is held fixed. In the second situation, k is usually fixed by utilizing a bounded slab of plasma of thickness $\frac{2\pi}{k}$. Thus, it is physically more meaningful to talk of standing waves in such a bounded plasma, and a calculation was carried out for this case.[4] Recognizing that our first case is probably more significant for traveling waves, we shall nevertheless list both results for the sake of comparison.

In the low frequency limit specified above, the result for the traveling wave becomes

$$\frac{\Delta k}{k} \cong \frac{1}{4}\left(\frac{b}{B_0}\right)^2\left(\frac{kV}{\Omega}\right)^2 \tag{2}$$

for fixed ω, and

$$\frac{\Delta\omega}{\omega} \cong -\frac{1}{2}\left(\frac{b}{B_0}\right)^2\left(\frac{kV}{\Omega}\right)^2 \tag{3}$$

for fixed k, where b and B_0 are the wave and unperturbed magnetic fields respectively and V is the electron thermal velocity.

We shall also show as a check that consideration of the cyclotron dispersion relation (21) leads in a natural way to the correct sign for $\frac{\Delta k}{k}$ and $\frac{\Delta\omega}{\omega}$, and to the correct ratio of their magnitudes (i.e., $\left|\frac{\Delta\omega}{\omega}\right| \Big/ \left|\frac{\Delta k}{k}\right| = 2$).

The second example involving wave effects in a "magnetized" plasma considers internally excited longitudinal electrostatic oscillations propagating at arbitrary angles with respect to the magnetic field. We find that collisions between these plasma oscillations can produce electromagnetic radiation that, for some electron velocity distributions, is split by the magnetic field. It is believed that this mechanism is responsible for the observed structure of Type II solar radio bursts (see Figure VIII). In this problem we are dealing directly with a Vlasov plasma and kinetic theory is employed as opposed to the macroscopic approach of the first example.

A flux of energetic (superthermal) electrons coexisting with a thermal background can, under some circumstances, generate very high amplitude longitudinal waves.[5] Basically, the energetic electrons emit electrostatic waves by a process of Cerenkov emission. If Landau damping is very small for certain ranges of wave phase velocity, then the wave or fluctuation spectrum for this range can become very large compared with thermal fluctuations. The limiting amplitude for this process is determined by the balance between reabsorption of the longitudinal waves by Landau damping and the Cerenkov emission by the superthermal electrons. These longitudinal electron plasma waves convert a part of their energy into electromagnetic radiation with frequency ω_e by scattering off ion plasma waves, and a part into radiation at $2\omega_e$ by scattering off each other, where ω_e is the electron plasma frequency $(4\pi e^2 n_0/m)^{\frac{1}{2}}$ (Tidman and Weiss,[6,7] Sturrock,[8,9] Cohen[10,11,12]). Thus we see that radiation from such a plasma can be enhanced many orders of magnitude over that generated by thermal fluctuations in a quiescent plasma.

It has been suggested by many authors (for a review of the literature see Wild, Smerd, and Weiss[13]) that such a radiation process may be the mechanism involved in Type II solar radio bursts (Ginzburg and Zheleznyakov[14]), although there are various ideas on how the electron plasma oscillations are excited.

The purpose of the second section of this work is to calculate the effects of a weak magnetic field on the plasma radiation spectrum, since it has long been suspected that such a field may be responsible for the observed line splitting in some Type II events. The only significant change in lowest order due to a weak magnetic field is in the real part of the Landau denominator $\mathscr{E}(\underline{k}, \omega)$. That is, the roots of the equation $\mathrm{Re}(\mathscr{E}) = 0$ change with the inclusion of the field. For small wave-number electron plasma waves the dispersion relation now becomes $\omega^2 \cong \omega_e^2 + \Omega^2 \sin^2\theta$ where θ is the angle between the magnetic field and the propagation vector $\underline{k}$ for the longitudinal wave and Ω is the electron cyclotron frequency $\frac{eB_0}{mc}$ (see Figure I). Thus we see that the frequency of these longitudinal electrostatic waves changes with their direction of propagation. Ion waves on the other hand propagate with frequencies approximately zero compared to ω_e for all directions of propagation.

If we now consider electron distributions which concentrate the propagation vectors in the directions parallel and perpendicular to the magnetic field, then we can generate electromagnetic radiation which is split in frequency. This process can be represented schematically as follows:

$$\begin{aligned}
&(\mathrm{epo}, \omega_e)_{\parallel} + (\mathrm{ipo}, 0)_{\parallel} \rightarrow (\mathrm{EM}, \omega_e) \\
&\left(\mathrm{epo}, \omega_e + \frac{\Omega^2}{2\omega_e}\right)_{\perp} + (\mathrm{ipo}, 0)_{\perp} \rightarrow \left(\mathrm{EM}, \omega_e + \frac{\Omega^2}{2\omega_e}\right) \\
&(\mathrm{epo}, \omega_e)_{\parallel} + (\mathrm{epo}, \omega_e)_{\parallel} \rightarrow (\mathrm{EM}, 2\omega_e) \\
&\left(\mathrm{epo}, \omega_e + \frac{\Omega^2}{2\omega_e}\right)_{\perp} + \left(\mathrm{epo}, \omega_e + \frac{\Omega^2}{2\omega_e}\right)_{\perp} \rightarrow \left(\mathrm{EM}, 2\omega_e + \frac{\Omega^2}{\omega_e}\right)
\end{aligned} \tag{4}$$

where epo, ipo, and EM denote electron plasma oscillations, ion plasma oscillations, and electromagnetic radiation respectively. The approximate frequencies of these waves are given in the second argument of each bracket with subscripts denoting parallel or perpendicular propagation of the longitudinal waves. Thus we see that the splitting of the second harmonic is twice that for the fundamental, in agreement with observations by Roberts.[15]

These electromagnetic waves are generated by collisions which are almost head-on between longitudinal waves of almost equal wavelengths. This may be represented symbolically by terms such as

$$(\omega - \omega', \underline{k})_{epo} + (\omega', \underline{K} - \underline{k})_{epo} = (\omega, \underline{K})_{EM} \; . \tag{5}$$

Since $K \cong 0\left(\frac{k_D V_e}{c}\right) << k$, we may approximate by

$$(\omega - \omega', \underline{k})_{epo} + (\omega', -\underline{k})_{epo} \cong (\omega, \underline{K})_{EM} \; , \tag{6}$$

as is illustrated in Figure II.

Specific superthermal electron distributions which concentrate the propagation vectors along the desired directions are found to be those with a kinetic temperature perpendicular to $\underline{B}_0$ much larger than that parallel to $\underline{B}_0$, which also have a net drift through the background plasma along $\underline{B}_0$. The longitudinal wave spectrum perpendicular to $\underline{B}_0$ is enhanced by electrostatic Cerenkov emission by the energetic electrons, and that parallel to $\underline{B}_0$ by the streaming motion which might possibly verge on two-stream instability for some wavenumbers.

It will become clear from our calculations that the splitting depends sensitively upon the specific distributions assumed. Isotropic velocity distributions simply broaden each emitted line, without generating a splitting. Other distributions, while leading to a mathematical splitting, may not yield enough energy in one part of the line for the splitting to be observable.

It is believed that these types of anisotropic distributions are physically plausible and might exist in a Type II radio burst. However, the main part of the calculations does not depend specifically on this assumption and the application to Type II events is therefore considered separately.

CHAPTER II

CYCLOTRON WAVES IN A 'HOT' PLASMA

We shall consider a collisionless electron gas coexisting with a uniform immobile positive ion background and a uniform magnetic field. We take velocity moments of the Vlasov Equation and allow for thermal effects by the inclusion of a pressure tensor. However, in the interests of obtaining frequency and wavenumber shifts which are calculable, we shall neglect the gradient of the heat flow tensor and thereby close our moment equations.

A. Basic Equations

Our basic equations are, therefore,

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x} \cdot (N\underline{V}) = 0 \tag{7}$$

$$\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \frac{\partial}{\partial \underline{x}} \underline{V} + \frac{1}{mN} \frac{\partial}{\partial \underline{x}} \cdot \underline{\underline{p}} + \frac{e}{m}\left(\underline{E} + \frac{1}{c}\underline{V} \times \underline{B}\right) = 0 \tag{8}$$

$$\frac{\partial \underline{\underline{p}}}{\partial t} + \underline{V} \cdot \frac{\partial}{\partial \underline{x}} \underline{\underline{p}} + \frac{e}{m}\left[\underline{\underline{p}} \times \underline{B} + \mathrm{tr}(\underline{\underline{p}} \times \underline{B})\right]$$

$$+ \underline{\underline{p}} \cdot \frac{\partial}{\partial \underline{x}} \underline{V} + \mathrm{tr}\left(\underline{\underline{p}} \cdot \frac{\partial}{\partial \underline{x}} \underline{V}\right) + \underline{\underline{p}} \frac{\partial}{\partial \underline{x}} \cdot \underline{V} = 0 \tag{9}$$

together with Maxwell's equations

$$\frac{\partial}{\partial \underline{x}} \times \underline{B} = \frac{1}{c}\frac{\partial \underline{E}}{\partial t} - \frac{4\pi e N \underline{V}}{c}, \quad \frac{\partial}{\partial \underline{x}} \times \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t}, \quad \frac{\partial}{\partial \underline{x}} \cdot E = 4\pi e(n_0 - N) \tag{10}$$

$tr(\underline{\underline{A}})$ means the transpose of $\underline{\underline{A}}$, N is the number density, $\underline{V}$ the velocity, $\underline{\underline{p}}$ the pressure tensor, $\underline{E}$ and $\underline{B}$ the electric and magnetic fields, and n_0 the uniform positive ion density.

We now seek a Krylov-Bogoliubov-Mitropolsky perturbation expansion of the form

$$\begin{aligned}
\underline{B} &= \underline{B}_0 + \epsilon \underline{B}^{(0)}(a, \psi) + \epsilon^2 \underline{B}^{(1)}(a, \psi) + \epsilon^3 \underline{B}^{(2)}(a, \psi) + \ldots \\
\underline{V} &= \epsilon \underline{v}^{(0)}(a, \psi) + \epsilon^2 \underline{v}^{(1)}(a, \psi) + \ldots \\
N &= n_0 + \epsilon n^{(0)}(a, \psi) + \ldots \\
\underline{\underline{p}} &= P_0 \underline{\underline{I}} + \epsilon \underline{\underline{p}}^{(0)}(a, \psi) + \ldots \\
\underline{E} &= \epsilon \underline{E}^{(0)}(a, \psi) + \ldots
\end{aligned} \tag{11}$$

where the successive corrections are assumed to be functions of a single amplitude 'a' and phase variable 'ψ' for the wave. We are interested in developing the expansion for the case in which taking the limit $\epsilon \to 0$ recovers the right-circularly polarized traveling cyclotron wave

$$\left.\begin{aligned}
\underline{E}^{(0)} &= a\,(\cos(kz - \omega t), \ -\sin(kz - \omega t), \ 0) \\
\underline{B}_0 &= (0, \ 0, \ B_0)
\end{aligned}\right\} \tag{12}$$

The appropriate expansions for the phase and amplitude variables are

$$\frac{\partial\psi}{\partial t} = -\omega + \epsilon^2 A(a) + \ldots, \quad \frac{\partial a}{\partial t} = \epsilon^2 B(a) + \ldots,$$
$$\frac{\partial\psi}{\partial z} = k + \epsilon^2 C(a) + \ldots, \quad \frac{\partial a}{\partial z} = \epsilon^2 D(a) + \ldots . \tag{13}$$

The functions A, B, C, and D are to be determined by requiring that there be no secular (ψ proportional) terms[3] in the expansion scheme (11). It should be noted that the first corrections in (13) are $0(\epsilon^2)$ rather than $0(\epsilon)$. This is a consequence of the only nonlinear terms of $0(\epsilon^2)$ in (7)-(10) being quadratic in the perturbations. No secularity arises in the $0(\epsilon^2)$ equations.

B. $0(\epsilon)$ Equations

We substitute the expansions (11)-(13) in (7)-(10) and equate terms of $0(\epsilon)$ to obtain the equations for linear wave propagation in an electron plasma.

$$-\omega \frac{\partial n^{(0)}}{\partial\psi} + k n_0 \frac{\partial v_z^{(0)}}{\partial\psi} = 0 \tag{14}$$

$$-\omega \frac{\partial v_i^{(0)}}{\partial\psi} + \frac{k}{mn_0}\frac{\partial p_{iz}^{(0)}}{\partial\psi} + \frac{e}{m} E_i^{(0)} + \frac{e}{mc}(\underline{v}^{(0)} \times \underline{B}_0)_i = 0 \tag{15}$$

$$-\omega \frac{\partial p_{ij}^{(0)}}{\partial\psi} + P_0 \delta_{ij} k \frac{\partial v_z^{(0)}}{\partial\psi} + P_0 k \frac{\partial v_i^{(0)}}{\partial\psi} \delta_{zj} + P_0 k \frac{\partial v_j^{(0)}}{\partial\psi} \delta_{zi}$$
$$+ \frac{e}{mc} B_{0k}(\epsilon_{j\ell k}\, p_{\ell i}^{(0)} + \epsilon_{i\ell k}\, p_{\ell j}^{(0)}) = 0 \tag{16}$$

$$\frac{\partial}{\partial \underline{x}} \times \underline{E}^{(0)} = \frac{\omega}{c}\frac{\partial \underline{B}^{(0)}}{\partial\psi} \tag{17}$$

$$\underline{v}^{(0)} = -\frac{c}{4\pi e n_0}\left(\frac{\partial}{\partial \underline{x}} \times \underline{B}^{(0)} + \frac{\omega}{c}\frac{\partial \underline{E}^{(0)}}{\partial \psi}\right) \tag{18}$$

where ϵ_{ijk} is the anti-symmetric unit tensor. Here we are assuming that all quantities depend on the space variable z only. Where convenient, we have also used subscript notation for the vectors and tensors ($i, j = 1, 2, 3$ etc.).

This system has solutions corresponding to a traveling right circularly-polarized cyclotron wave

$$\left.\begin{aligned}
\underline{E}^{(0)} &= a(\cos\psi, \; -\sin\psi, \; 0) \\
\underline{B}^{(0)} &= \frac{cak}{\omega}(\sin\psi, \; \cos\psi, \; 0) \\
\underline{v}^{(0)} &= \frac{a(\omega^2 - c^2k^2)}{4\pi e n_0 \omega}(\sin\psi, \; \cos\psi, \; 0) \\
n^{(0)} &= 0 \\
\underline{\underline{p}}^{(0)} &= \begin{bmatrix} 0 & 0 & p_{xz}^{(0)} \\ 0 & 0 & p_{yz}^{(0)} \\ p_{zx}^{(0)} & p_{zy}^{(0)} & 0 \end{bmatrix}
\end{aligned}\right\} \tag{19}$$

where

$$\left.\begin{aligned}
p_{xz}^{(0)} = p_{zx}^{(0)} &= \frac{ak\,P_0(c^2k^2 - \omega^2)}{4\pi e n_0 \omega(\Omega - \omega)}\sin\psi \\
p_{yz}^{(0)} = p_{zy}^{(0)} &= \frac{akP_0(c^2k^2 - \omega^2)}{4\pi e n_0 \omega(\Omega - \omega)}\cos\psi
\end{aligned}\right\} \tag{20}$$

The frequency ω and wavenumber k are related by the dispersion relation for right circularly polarized cyclotron waves

$$c^2k^2 - \omega^2 - \frac{\omega\omega_e^2(\Omega - \omega)}{[(\Omega - \omega)^2 - k^2V^2]} = 0 \ , \tag{21}$$

and to $0(\epsilon)$, $\psi = kz - \omega t$. V is the thermal velocity defined by

$$V^2 = P_0/mn_0 \tag{22}$$

For very low frequencies, $\omega^2 << \Omega^2 >> k^2V^2$, $\omega^2 << c^2k^2$, (21) reduces to the helicon dispersion relation

$$\omega \cong c^2k^2\Omega/\omega_e^2 \ . \tag{23}$$

Note that if we had started with a <u>left</u>-circularly polarized wave

$$\underline{E}^{(0)} = a(\cos\psi, \ \sin\psi, \ 0) \tag{24}$$

we would have obtained a dispersion relation

$$c^2k^2 - \omega^2 + \frac{\omega\omega_e^2(\Omega + \omega)}{[(\Omega + \omega)^2 - k^2V^2]} = 0 \ . \tag{25}$$

In the same low frequency limit we obtain $c^2k^2 \cong -\omega\omega_e^2/\Omega$, i.e., no propagation for this mode when $\omega << \Omega$.

C. First Order Corrections to the Linear Wave Solution; $0(\epsilon^2)$ Equations

We now proceed to the $0(\epsilon^2)$ equations in the perturbation expansion. Thus we substitute (11)-(13) in (7)-(10), make use of the $0(\epsilon)$ solutions (19) and pick out the $0(\epsilon^2)$ terms, namely,

$$-\omega \frac{\partial n^{(1)}}{\partial \psi} + n_0 k \frac{\partial v_z^{(1)}}{\partial \psi} = 0 \tag{26}$$

$$-\omega \frac{\partial v_i^{(1)}}{\partial \psi} + \frac{k}{mn_0} \frac{\partial p_{iz}^{(1)}}{\partial \psi} + \frac{e}{m} E_i^{(1)} + \frac{e}{mc} (\underline{v}^{(1)} \times \underline{B}^{(0)})_i$$

$$= -\frac{e}{mc} (\underline{v}^{(0)} \times \underline{B}^{(0)})_i \tag{27}$$

$= 0$ since $\underline{v}^{(0)}$ and $\underline{B}^{(0)}$ are antiparallel.

$$-\omega \frac{\partial p_{ij}^{(1)}}{\partial \psi} + \Omega[\epsilon_{j\ell z}\, p_{\ell i}^{(1)} + \epsilon_{i\ell z}\, p_{\ell j}^{(1)}]$$

$$+ P_0 k \left[\frac{\partial v_z^{(1)}}{\partial \psi} \delta_{ij} + \frac{\partial v_i^{(1)}}{\partial \psi} \delta_{jz} + \frac{\partial v_j^{(1)}}{\partial \psi} \delta_{iz}\right]$$

$$= -\frac{e}{mc} B_k^{(0)} [\epsilon_{j\ell k}\, p_{\ell i}^{(0)} + \epsilon_{i\ell k}\, p_{\ell j}^{(0)}]$$

$$- p_{jz}^{(0)} k \frac{\partial v_i^{(0)}}{\partial \psi} - p_{iz}^{(0)} k \frac{\partial v_j^{(0)}}{\partial \psi} \tag{28}$$

$$\frac{\partial}{\partial \underline{x}} \times \underline{B}^{(1)} + \frac{\omega}{c} \frac{\partial \underline{E}^{(1)}}{\partial \psi} + \frac{4\pi e n_0 \underline{v}^{(1)}}{c} = 0 \tag{29}$$

$$\frac{\partial}{\partial \underline{x}} \times \underline{E}^{(1)} - \frac{\omega}{c} \frac{\partial \underline{B}^{(1)}}{\partial \psi} = 0 \tag{30}$$

$$k \frac{\partial E_z^{(1)}}{\partial \psi} + 4\pi e n^{(1)} = 0 \ . \tag{31}$$

These equations are linear in the first corrections $\underline{E}^{(1)}$, etc., to the wave variables, but involve source terms on the right hand side which are quadratic in the $0(\epsilon)$ variables.

Using the lowest-order solutions (19), we can write out the various components of Eq. (28) for $p_{ij}^{(1)}$ as follows:

$$-\omega \frac{\partial p_{yz}^{(1)}}{\partial \psi} - \Omega p_{xz}^{(1)} + P_0 k \frac{\partial v_y^{(1)}}{\partial \psi} = 0 \tag{32}$$

$$-\omega \frac{\partial p_{xz}^{(1)}}{\partial \psi} + \Omega p_{yz}^{(1)} + P_0 k \frac{\partial v_x^{(1)}}{\partial \psi} = 0 \tag{33}$$

$$-\omega \frac{\partial p_{zz}^{(1)}}{\partial \psi} + 3P_0 k \frac{\partial v_z^{(1)}}{\partial \psi} = 0 \tag{34}$$

$$-\omega \frac{\partial p_{xy}^{(1)}}{\partial \psi} + \Omega p_{yy}^{(1)} - \Omega p_{xx}^{(1)} = -\frac{e}{mc}[B_x^{(0)} p_{zx}^{(0)} - B_y^{(0)} p_{zy}^{(0)}]$$

$$- p_{zx}^{(0)} k \frac{\partial v_y^{(0)}}{\partial \psi} - p_{zy}^{(0)} k \frac{\partial v_x^{(0)}}{\partial \psi} \tag{35}$$

(35) becomes

$$-\omega \frac{\partial p_{xy}^{(1)}}{\partial \psi} + \Omega p_{yy}^{(1)} - \Omega p_{xx}^{(1)} = \frac{a^2 k^2 P_0 (c^2 k^2 - \omega^2)^2}{16\pi^2 e^2 n_0^2 \omega^3 (\Omega - \omega)^2} \cdot$$

$$\cdot [(\Omega - \omega)\Omega - k^2 V^2] \cos 2\psi \equiv X \cos 2\psi , \tag{36}$$

$$-\omega \frac{\partial p_{yy}^{(1)}}{\partial \psi} - 2\Omega p_{xy}^{(1)} + P_0 \frac{\partial v_z^{(1)}}{\partial \psi} = -\frac{2e}{mc} B_x^{(0)} p_{zy}^{(0)}$$

$$- 2p_{zy}^{(0)} k \frac{\partial v_y^{(0)}}{\partial \psi} = - X \sin 2\psi , \tag{37}$$

$$-\omega \frac{\partial p_{xx}^{(1)}}{\partial \psi} + 2\Omega p_{xy}^{(1)} + P_0 \frac{\partial v_z^{(1)}}{\partial z} = X \sin 2\psi . \tag{38}$$

We are concerned only with the inhomogeneous part of the solution. To this one could add, of course, any solution of the homogeneous equations in order to satisfy various initial conditions in the excitation of the oscillations. However (see Ref. 4) only the inhomogeneous part of the solution contributes to the frequency and wavenumber shifts obtained in $0(\epsilon^3)$. Thus, noting that to $0(\epsilon^2)$ no secularity occurs, we find,

$$\left.\begin{aligned} &\underline{v}^{(1)} = \underline{B}^{(1)} = \underline{E}^{(1)} = n^{(1)} = 0 \\ &\underline{\underline{p}}^{(1)} = \begin{bmatrix} p_{xx}^{(1)} & p_{xy}^{(1)} & 0 \\ p_{yz}^{(1)} & p_{yy}^{(1)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}\right\} \tag{39}$$

where
$$p_{xy}^{(1)} = p_{yx}^{(1)} = -\frac{X \sin 2\psi}{2(\omega - \Omega)}, \tag{40}$$

and
$$p_{xx}^{(1)} = -p_{yy}^{(1)} = \frac{X \cos 2\psi}{2(\omega - \Omega)}, \tag{41}$$

and X is defined in Equation (36).

We note from these expressions that the perturbation expansion is valid only if we can treat $(\Omega - \omega)$ as an $0(1)$ quantity.

The solutions (39)-(41) represent the first correction to the strictly linear wave solutions (19)-(20). In this order no secular terms have arisen and the corrections are identical with those one would obtain with conventional perturbation theory. It is in $0(\epsilon^3)$ that we find secularity and a finite wavenumber or frequency shift. Note that these corrections (39)-(41) in the traveling wave case involve only the pressure tensor. Thus a traveling wave in a 'cold' plasma would clearly not generate any such wavenumber or frequency shifts.

D. Second-Order Corrections to the Linear Wave Solution; $0(\epsilon^3)$ Equations

We substitute the expansions (11)-(13) into (7)-(10) and pick out the $0(\epsilon^3)$ terms to obtain the equations for the second order corrections to the cyclotron wave solutions. They are

$$-\omega \frac{\partial n^{(2)}}{\partial\psi} + k n_0 \frac{\partial v_z^{(2)}}{\partial\psi} = 0 \tag{42}$$

$$-\omega \frac{\partial v_i^{(2)}}{\partial\psi} + \left(A\frac{\partial}{\partial\psi} + B\frac{\partial}{\partial a}\right) v_i^{(0)} + \frac{e}{m} E_i^{(2)} + \frac{k}{mn_0}\frac{\partial p_{iz}^{(2)}}{\partial\psi}$$
$$+ \frac{1}{mn_0}\left(C\frac{\partial}{\partial\psi} + D\frac{\partial}{\partial a}\right) p_{iz}^{(0)} + \frac{e}{mc}(\underline{v}^{(2)} \times \underline{B}_0)_i = 0 \tag{43}$$

$$-\omega \frac{\partial p_{ij}^{(2)}}{\partial\psi} + \left(A\frac{\partial}{\partial\psi} + B\frac{\partial}{\partial a}\right) p_{ij}^{(0)} + \frac{e}{mc} B_{0k}\,[\epsilon_{j\ell k}\, p_{\ell i}^{(2)} + \epsilon_{i\ell k}\, p_{\ell j}^{(2)}]$$
$$+ \frac{eB}{mc} k^{(0)} [\epsilon_{j\ell k}\, p_{\ell i}^{(1)} + \epsilon_{i\ell k}\, p_{\ell j}^{(1)}] + P_0\, \delta_{jz} k \frac{\partial v_i^{(2)}}{\partial\psi} + P_0\, \delta_{iz} k \frac{\partial v_j^{(2)}}{\partial\psi}$$
$$+ P_0\, k\, \delta_{ij} \frac{\partial v_z^{(2)}}{\partial\psi} + P_0\, \delta_{jz}\left(C\frac{\partial}{\partial\psi} + D\frac{\partial}{\partial a}\right) v_i^{(0)}$$
$$+ P_0\, \delta_{iz}\left(C\frac{\partial}{\partial\psi} + D\frac{\partial}{\partial a}\right) v_j^{(0)} = 0 \tag{44}$$

$$\underline{k} \times \frac{\partial \underline{E}^{(2)}}{\partial \psi} + \left(C \frac{\partial}{\partial \psi} + D \frac{\partial}{\partial a} \right) \hat{e}_z \times \underline{E}^{(0)}$$

$$= -\frac{1}{c} \left\{ -\omega \frac{\partial \underline{B}^{(2)}}{\partial \psi} + \left(A \frac{\partial}{\partial \psi} + B \frac{\partial}{\partial a} \right) \underline{B}^{(0)} \right\} \tag{45}$$

$$\underline{k} \times \frac{\partial \underline{B}^{(2)}}{\partial \psi} + \left(C \frac{\partial}{\partial \psi} + D \frac{\partial}{\partial a} \right) \hat{e}_z \times \underline{B}^{(0)}$$

$$= \frac{1}{c} \left\{ -\omega \frac{\partial \underline{E}^{(2)}}{\partial \psi} + \left(A \frac{\partial}{\partial \psi} + B \frac{\partial}{\partial a} \right) \underline{E}^{(0)} \right\} - \frac{4\pi e}{c} \left\{ n_0 \underline{v}^{(2)} \right\} \tag{46}$$

$$k \frac{\partial E_z^{(2)}}{\partial \psi} = -4\pi e n^{(2)} \quad . \tag{47}$$

When we solve for $\underline{E}^{(2)}$, $p_{ij}^{(2)}$, ..., etc., we find secular (ψ-proportional) terms. We eliminate them by a proper choice of A, B, C, and D. We now wish to obtain an equation that involves only $\frac{\partial \underline{\underline{v}}^{(2)}}{\partial \psi}$ in its linear part, and has a source term involving the perturbations $p_{ij}^{(1)}$, etc., and $p_{ij}^{(0)}$, etc.

First, consider the equations for the quantities $p_{iz}^{(2)}$ which follow from (44). They are,

$$-\frac{\partial p_{zz}^{(2)}}{\partial \psi} = -3 P_0 k \frac{\partial v_z^{(2)}}{\partial \psi} \tag{48}$$

$$-\omega \frac{\partial p_{xz}^{(2)}}{\partial \psi} + \Omega p_{yz}^{(2)} = \Lambda \equiv -P_0 k \frac{\partial v_x^{(2)}}{\partial \psi} - \left(A \frac{\partial}{\partial \psi} + B \frac{\partial}{\partial a} \right) p_{xz}^{(0)}$$

$$-\frac{e}{mc} B_k^{(0)} \epsilon_{z\ell k} p_{\ell x}^{(1)} - P_0 \left(C \frac{\partial}{\partial \psi} + D \frac{\partial}{\partial a} \right) v_x^{(0)} = -P_0 k \frac{\partial v_x^{(2)}}{\partial \psi} + b(\psi) \tag{49}$$

$$-\omega \frac{\partial p_{yz}^{(2)}}{\partial\psi} - \Omega p_{xz}^{(2)} = \Gamma \equiv -P_0 k \frac{\partial v_y^{(2)}}{\partial\psi} - \left(A \frac{\partial}{\partial\psi} + B \frac{\partial}{\partial a}\right) p_{yz}^{(0)}$$

$$-\frac{e}{mc} B_k^{(0)} \epsilon_{z\ell k}\, p_{\ell y}^{(1)} - P_0 \left(C \frac{\partial}{\partial\psi} + D \frac{\partial}{\partial a}\right) v_y^{(0)} = -P_0 k \frac{\partial v_y^{(2)}}{\partial\psi} + d(\psi) \quad (50)$$

Taking $\frac{\partial}{\partial\psi}$ of (49) and using (50), or taking $\frac{\partial}{\partial\psi}$ of (50) and using (49) gives

$$\left.\begin{aligned} \left(\omega^2 \frac{\partial^2}{\partial\psi^2} + \Omega^2\right) p_{xz}^{(2)} &= -\Omega\Gamma - \omega \frac{\partial\Lambda}{\partial\psi} \\ \left(\omega^2 \frac{\partial^2}{\partial\psi^2} + \Omega^2\right) p_{yz}^{(2)} &= \Omega\Lambda - \omega \frac{\partial\Gamma}{\partial\psi} \end{aligned}\right\} \quad (51)$$

Next, we operate with $\left(\omega^2 \frac{\partial^2}{\partial\psi^2} + \Omega^2\right)$ on (43) and use (51) to eliminate $p_{xz}^{(2)}$ and $p_{yz}^{(2)}$. We also make use of (45) and (46) to eliminate $\underline{B}^{(2)}$ and $\underline{E}^{(2)}$. Similarly, we use (42), (43), (44) and (47) to find an equation for $\frac{\partial v_z^{(2)}}{\partial\psi}$. We thereby find an equation which can be written as

$$\underline{\underline{L}} \cdot \frac{\partial \underline{v}^{(2)}}{\partial\psi} = \underline{S} \quad (52)$$

$$\text{where } \underline{\underline{L}} = \begin{bmatrix} a_{xx} & a_{xy} & 0 \\ a_{yx} & a_{yy} & 0 \\ 0 & 0 & a_{zz} \end{bmatrix} ; \quad \underline{S} = \begin{bmatrix} s_x \\ s_y \\ 0 \end{bmatrix} \quad (53)$$

with

$$a_{xx} = a_{yy} = -\omega \left\{ (\omega^2 - k^2V^2) \frac{\partial^4}{\partial\psi^4} + \left(\Omega^2 - \frac{\omega^2 \omega_e^2}{c^2k^2 - \omega^2}\right) \frac{\partial^2}{\partial\psi^2} - \frac{\omega_e^2 \Omega^2}{c^2k^2 - \omega^2} \right\} \quad (54)$$

$$a_{xy} = -a_{yx} = \Omega \left\{ (\omega^2 + k^2V^2) \frac{\partial^3}{\partial\psi^3} + \Omega^2 \frac{\partial}{\partial\psi} \right\} \quad (55)$$

$$a_{zz} = (\omega^2 - 3k^2V^2)\frac{\partial^2}{\partial\psi^2} - \omega_e^2 \tag{56}$$

and

$$s_x = \frac{k}{mn_0}\left\{\Omega\frac{\partial^3 d}{\partial\psi^3} + \omega\frac{\partial^4 b}{\partial\psi^4}\right\} - \left(\omega^2\frac{\partial^2}{\partial\psi^2} + \Omega^2\right)\left\{\left(A\frac{\partial}{\partial\psi} + B\frac{\partial}{\partial a}\right)\frac{\partial^2 v_x^{(0)}}{\partial\psi^2}\right.$$

$$\left. + \frac{1}{mn_0}\left(C\frac{\partial}{\partial\psi} + D\frac{\partial}{\partial a}\right)\frac{\partial^2 p_{xz}^{(0)}}{\partial\psi^2} - \frac{\omega_e^2\,\omega}{(c^2k^2 - \omega^2)}\frac{\partial\beta}{\partial\psi}\right\} \tag{57}$$

$$s_y = \frac{k}{mn_0}\left(\omega\frac{\partial^4 d}{\partial\psi^4} - \Omega\frac{\partial^3 b}{\partial\psi^3}\right) - \left(\omega^2\frac{\partial^2}{\partial\psi^2} + \Omega^2\right)\left\{\left(A\frac{\partial}{\partial\psi} + B\frac{\partial}{\partial a}\right)\frac{\partial^2 v_y^{(0)}}{\partial\psi^2}\right.$$

$$\left. + \frac{1}{mn_0}\left(C\frac{\partial}{\partial\psi} + D\frac{\partial}{\partial a}\right)\frac{\partial^2 p_{yz}^{(0)}}{\partial\psi^2} - \frac{\omega_e^2\,\omega}{(c^2k^2 - \omega^2)}\frac{\partial\gamma}{\partial\psi}\right\} \tag{58}$$

Recall from (49),

$$b = -\left(A\frac{\partial}{\partial\psi} + B\frac{\partial}{\partial a}\right)p_{xz}^{(0)} - \frac{e}{mc}B_k^{(0)}\ {}_{z\ell k}\,p_{\ell x} - P_0\left(C\frac{\partial}{\partial\psi} + D\frac{\partial}{\partial a}\right)v_x^{(0)}, \tag{59}$$

and from (50),

$$d = -\left(A\frac{\partial}{\partial\psi} + B\frac{\partial}{\partial a}\right)p_{yz}^{(0)} - \frac{e}{mc}B_k^{(0)}\epsilon_{z\ell k}\,p_{\ell y}^{(1)} - P_0\left(C\frac{\partial}{\partial\psi} + D\frac{\partial}{\partial a}\right)v_y^{(0)}. \tag{60}$$

In (57) and (58),

$$4\pi e n_0\beta = \left\{\left(A\frac{\partial}{\partial\psi} + B\frac{\partial}{\partial a}\right) + \frac{c^2k}{\omega}\left(C\frac{\partial}{\partial\psi} + D\frac{\partial}{\partial a}\right)\right\}E_x^{(0)}$$

$$+ c\left\{\frac{k}{\omega}\left(A\frac{\partial}{\partial\psi} + B\frac{\partial}{\partial a}\right) + \left(C\frac{\partial}{\partial\psi} + D\frac{\partial}{\partial a}\right)\right\}B_y^{(0)}, \tag{61}$$

and

$$4\pi e n_0 \gamma = \left\{ \left(A \frac{\partial}{\partial\psi} + B \frac{\partial}{\partial a} \right) + \frac{c^2 k}{\omega} \left(C \frac{\partial}{\partial\psi} + D \frac{\partial}{\partial a} \right) \right\} E_y^{(0)}$$

$$- c \left\{ \frac{k}{\omega} \left(A \frac{\partial}{\partial\psi} + B \frac{\partial}{\partial a} \right) + \left(C \frac{\partial}{\partial\psi} + D \frac{\partial}{\partial a} \right) \right\} B_x^{(0)} . \qquad (62)$$

We note from (52)-(53) that $\partial v_z^{(2)}/\partial\psi$ is decoupled from $\partial v_x^{(2)}/\partial\psi$ and $\partial v_y^{(2)}/\partial\psi$. Setting $\partial v_z^{(2)}/\partial\psi = F \cos n\psi$, for example, where n is an integer, we obtain

$$F[n^2(3k^2V^2 - \omega^2) - \omega_e^2] = 0 \qquad (63)$$

In (63), F must be set equal to zero, because the bracket does not satisfy the dispersion relation (21) for any n (i.e., it is non-zero).

Since $a_{xx} a_{xy} G(\psi) = a_{xy} a_{xx} G(\psi)$, where G is an arbitrary function of ψ, we are free to rewrite (52) as

$$(a_{xx}^2 + a_{xy}^2) \frac{\partial v_x^{(2)}}{\partial\psi} = a_{xx} s_x - a_{xy} s_y \equiv \mathscr{S}_x \qquad (64)$$

$$(a_{xx}^2 + a_{xy}^2) \frac{\partial v_y^{(2)}}{\partial\psi} = a_{xy} s_x + a_{xx} s_y \equiv \mathscr{S}_y \qquad (65)$$

The secular behavior of $\partial v_x^{(2)}/\partial\psi$ and $\partial v_y^{(2)}/\partial\psi$ is controlled by the right hand side of (64) and (65) respectively. Hence we will be able to eliminate this secularity by adjusting A, B, C, and D in $\mathscr{S}_x$ and $\mathscr{S}_y$. Consideration of either $\mathscr{S}_x$ or $\mathscr{S}_y$ alone will then give us our wave-number shift.

We expand $\dfrac{\partial \underline{v}^{(2)}}{\partial\psi}$ as a series of traveling right- and left-circularly polarized waves:

$$\frac{\partial \underline{v}^{(2)}}{\partial \psi} = \sum_{n=1}^{\infty} \{r_n(\cos n\psi, -\sin n\psi, 0) + \ell_n(\cos n\psi, \sin n\psi, 0)\} \tag{66}$$

If we pick out the fundamental component corresponding to the right-circularly polarized wave as in (19), i.e.,

$$\frac{\partial \underline{v}^{(2)}}{\partial \psi} \rightarrow \frac{\partial \underline{v}_R^{(2)}}{\partial \psi} = r_1(\cos \psi, -\sin \psi, 0) , \tag{67}$$

then it is easy to verify that

$$\left.\begin{aligned} a_{xx}(\cos \psi) + a_{xy}(-\sin \psi) = 0 \\ a_{yx}(\cos \psi) + a_{yy}(-\sin \psi) = 0 \end{aligned}\right\} . \tag{68}$$

Thus, (67) is a solution of the homogeneous part of (52), provided that ω and k satisfy the dispersion relation (21). The components $\cos \psi$ in $\mathscr{S}_x$ and $-\sin \psi$ in $\mathscr{S}_y$ will, therefore, contribute to secular behavior in $\frac{\partial v_x^{(2)}}{\partial \psi}$ and $\frac{\partial v_y^{(2)}}{\partial \psi}$ respectively, unless they are eliminated.

It turns out that $\mathscr{S}_x$ and $\mathscr{S}_y$ each contains $\cos \psi$ <u>and</u> $\sin \psi$ terms. The extra terms are brought in through the $\partial/\partial a$ operations in s_x and s_y which contribute $\sin \psi$ in $\mathscr{S}_x$ and $-\cos \psi$ in $\mathscr{S}_y$. Since it is easy to show that

$$\left.\begin{aligned} a_{xx}(\sin \psi) + a_{xy}(-\cos \psi) \neq 0 \\ a_{yx}(\sin \psi) + a_{yy}(-\cos \psi) \neq 0 \end{aligned}\right\} , \tag{69}$$

that is, $(\sin \psi, -\cos \psi, 0)$ is <u>not</u> a solution to the homogeneous part of (52), these extra terms are not involved in the secular behavior of

$\partial \underline{v}^{(2)}/\partial\psi$. We are free, therefore, to set $B = D = 0$ (that is, regard the amplitude 'a' as fixed) which will simplify the calculation of $\mathscr{S}_x$ and $\mathscr{S}_y$.

To determine the actual frequency or wavenumber shift, we need work with only one quantity, say $\mathscr{S}_x$, and require that its $\cos\psi$-proportional part vanish. This involves some straightforward but laborious algebra which consists of substituting $v_i^{(0)}$, $p_{iz}^{(0)}$, $\underline{B}^{(0)}$, $\underline{E}^{(0)}$, and $p_{ij}^{(1)}$ $(ij \neq z)$ into (57) to (62), and in (64) to obtain $\mathscr{S}_x$.

We find that s_x and s_y take the form

$$\begin{aligned} s_x &= K_1 \cos\psi \\ s_y &= -K_1 \sin\psi \end{aligned} \tag{70}$$

Thus,

$$\mathscr{S}_x = a_{xx} s_x - a_{xy} s_y = K_1 \{a_{xx}(\cos\psi) - a_{xy}(-\sin\psi)\} , \tag{71}$$

which can be written as

$$\mathscr{S}_x = K_1 \cos\psi \left\{ -\omega \left[\omega^2 - k^2V^2 - \Omega^2 + \frac{\omega_e^2(\omega^2 - \Omega^2)}{c^2k^2 - \omega^2} \right] - \Omega[\omega^2 + k^2V^2 - \Omega^2] \right\} . \tag{72}$$

In this case, we see that $\mathscr{S}_x$ contains only $\cos\psi$-proportional terms, and we simply set it equal to zero. Referring to the dispersion relation (21), we see that the curly bracket in (72) is non-zero, hence, K_1 must vanish.

We may note that had we used the non-resonant $\partial/\partial a$ part of $\mathscr{S}_x$ (i.e., the vector $(\sin\psi, -\cos\psi, 0)$) we would have obtained

$$\mathscr{S}_x = K_2 \sin\psi \left\{-\omega\left[\omega^2 - k^2V^2 - \Omega^2 + \frac{\omega_e^2(\omega^2-\Omega^2)}{c^2k^2-\omega^2}\right] + \Omega[\omega^2 + k^2V^2 - \Omega^2]\right\} = 0 \tag{73}$$

Since the bracket in (73) is now identically zero, we would be unable to determine K_2, thus confirming our earlier decision to set $B = D = 0$.

The condition $K_1 = 0$, therefore, will determine our wavenumber or frequency shift.

E. Wavenumber and Frequency Shifts

We find, for ω fixed (i.e., $A = 0$),

$$\frac{C}{k} = \frac{\Delta k}{k} = -\frac{a^2k^2k^2V^2(\Omega+\omega)(c^2k^2-\omega^2)\omega_e^2}{32\pi^2e^2n_0^2\omega^3(\Omega-\omega)^2} \cdot$$

$$\cdot(\omega_e^2 + c^2k^2 - \omega^2)\left\{\frac{k^2V^2(\Omega+\omega)(c^2k^2-\omega^2)}{\omega} + (\Omega^2-\omega^2)\left[\frac{k^2V^2(c^2k^2-\omega^2)}{\omega(\Omega+\omega)} - \frac{2\omega_e^2c^2k^2}{(c^2k^2-\omega^2)}\right]\right\}^{-1} . \tag{74}$$

We can simplify this expression considerably by going to the low frequency region appropriate to cyclotron (helicon) waves in a solid-state plasma:

$$\omega_e^2 \gg k^2c^2 \gg \Omega^2 \gg k^2V^2 \quad \text{where } \Omega^2 \gg \omega^2 , \tag{75}$$

and we may use our helicon dispersion relation (23).

Thus we obtain

$$\frac{\Delta k}{k} \cong \frac{1}{4}\left(\frac{b}{B_0}\right)^2\left(\frac{kV}{\Omega}\right)^2 \tag{76}$$

Similarly, for k fixed $(C = 0)$, we find

$$\frac{\Delta\omega}{\omega} = -\frac{A}{\omega} = -\frac{a^2 k^2 k^2 V^2 (\Omega+\omega)(c^2 k^2 - \omega^2)\,\omega_e^2\,(\omega_e^2 + c^2 k^2 - \omega^2)}{32\pi^2 e^2 n_0^2 \omega^3 (\Omega - \omega)^2} \cdot$$

$$\left\{ k^2 V^2 \left(\frac{\Omega+\omega}{\Omega-\omega}\right)(c^2 k^2 - \omega^2) + (\Omega^2 - \omega^2)\left[c^2 k^2 - \omega^2 + \omega_e^2 \left(\frac{c^2 k^2 + \omega^2}{c^2 k^2 - \omega^2}\right)\right] \right\}^{-1} . \tag{77}$$

In the same low frequency limit (75), we have

$$\frac{\Delta\omega}{\omega} \cong -\frac{1}{2}\left(\frac{b}{B_0}\right)^2 \left(\frac{kV}{\Omega}\right)^2 \tag{78}$$

These results differ with the case for the standing wave in a bounded slab of plasma[4] where stringent boundary conditions fix k. Here

$$\frac{\Delta\omega}{\omega} \cong \frac{3}{8}\left(\frac{b}{B_0}\right)^2 \left(\frac{kV}{\Omega}\right)^2 \quad . \tag{79}$$

Following the suggestion of R. Goldman,[16] we can check our results (76) and (78) by appealing to the cyclotron dispersion relation (21). We can thereby show that wavenumber and frequency shifts enter in a natural way. Instead of regarding (21) as exact, we now permit the finite amplitude effects to manifest themselves in the form of a modified dispersion relation.

Thus, we have originally

$$c^2 k^2 - \omega^2 - \frac{\omega\,\omega_e^2\,(\Omega - \omega)}{[(\Omega-\omega)^2 - k^2 V^2]} = 0 \quad . \tag{21}$$

In our standard low frequency limit, this can be written as

$$c^2k^2 - \frac{\omega\,\omega_e^2\,\Omega}{(\Omega^2 - k^2V^2)} \cong 0 \;, \tag{80}$$

or

$$k^2V^2 \cong \Omega^2 - \frac{\omega\,\omega_e^2\,\Omega}{c^2k^2} \tag{81}$$

Now we assume that all finite amplitude effects enter through the pressure terms and we replace V^2 by $\frac{p}{mn_0}$ to get

$$\frac{p}{mn_0} \cong \frac{\Omega^2}{k^2} - \frac{\omega\,\omega_e^2\,\Omega}{c^2k^4} \tag{82}$$

The non-linearities are introduced by letting p become $p + \Delta p$, ω go to $\omega + \Delta\omega$ (for fixed k), and k go to $k + \Delta k$ (for fixed ω).

For fixed ω, we get

$$\frac{\Delta k}{k} \cong \frac{k^2}{2\Omega^2}\,\frac{\Delta p}{mn_0} \quad , \tag{83}$$

and for fixed k, we get

$$\frac{\Delta\omega}{\omega} \cong -\frac{k^2}{\Omega^2}\,\frac{\Delta p}{mn_0} \quad . \tag{84}$$

Thus, (83) and (84) have the correct sign and their ratio has the correct magnitude (see (76) and (78)).

CHAPTER III

LINE SPLITTING OF PLASMA RADIATION IN A WEAK MAGNETIC FIELD

We turn now to the second example of wave motion discussed in Chapter I where large amplitude longitudinal waves in a highly excited collisionless plasma combine, in the presence of a weak magnetic field, to produce electromagnetic radiation which is split in frequency.

A. Basic Equations and Spectral Densities

In the following calculations we use an expression derived by Dupree[17,18] for the emission of radiation by a plasma and reduced for frequencies near ω_e and $2\omega_e$ by Tidman and Dupree.[5] Similar formulas can also be derived by the technique of Birmingham, Dawson and Oberman.[19] We have, from equations (14), (15) and (16) of Ref. 5,

$$\left(\frac{dU}{dt}\right)_{\text{emiss}} = \left(\frac{dU}{dt}\right)_1 + \left(\frac{dU}{dt}\right)_2 + 0(K^4) \ , \tag{85}$$

$$\left(\frac{dU}{dt}\right)_1 = \frac{n_0^2 e^2 \omega_e^4}{2\pi\omega^2} \int_{-\infty}^{\infty} \frac{d\underline{k}}{(2\pi)^3} \int_{-\infty}^{\infty} d\omega' \frac{|\underline{k}\cdot\underline{\epsilon}_0|^2}{k^4} S_{ee}(\underline{k}, \omega') S_{ii}(-\underline{k}, \omega-\omega') \tag{86}$$

$$\left(\frac{dU}{dt}\right)_2 = \frac{n_0^2 e^2 \omega_e^4}{\pi\omega^2} \int_{-\infty}^{\infty} \frac{d\underline{k}}{(2\pi)^3} \int_{-\infty}^{\infty} d\omega' \frac{|\underline{k}\cdot\underline{\epsilon}_0|^2 (\underline{k}\cdot\underline{K})^2}{k^8} S_{ee}(\underline{k}, \omega') S_{ee}(-\underline{k}, \omega-\omega') \ , \tag{87}$$

where $\frac{dU}{dt}(\underline{K})$ is the rate at which the energy density $U(\underline{K})$ of a transverse mode with a propagation vector $\underline{K}$, frequency ω, and polarization vector $\underline{\epsilon}_0$ $(|\underline{\epsilon}_0|^2 = 2\pi)$, increases in the plasma. To obtain the actual emission intensity, dU/dt must be multiplied by the density of states $d^2n(\underline{K})/d\sigma\, d\omega$ where $d\sigma$ is an element of solid angle in the direction $\underline{K}$ and ω and $\underline{K}$ satisfy the dispersion relation for transverse waves.

We note that the emission of radiation (Equations (86) and (87)) is directly related to the spectral density $S_{\alpha\beta}(\underline{k}, \omega)$, etc., for the colliding electrostatic waves, and we will shortly see that this seems entirely reasonable. Whenever electrons in the plasma are accelerated by an electric field (in this case, electrostatic oscillations), they radiate by the process of bremsstrahlung. Both the incoming electric field and the outgoing radiation are affected by the collective properties of the plasma which manifest themselves through the dielectric function or Landau denominator $\mathscr{E}(\underline{k}, i\omega)$. As noted in Chapter I, it is primarily this quantity, through $\mathrm{Re}(\mathscr{E})$, which is modified by the inclusion of a magnetic field. If the electrons respond to incoming electric fields that fluctuate randomly they will radiate with random phases and the averaged emitted radiation will be negligible. The lack of randomness between two different electric fields or number density fluctuations can be expressed in terms of the spectral density. This is the Fourier transform of the autocorrelation function, which as its name suggests, attempts to measure the correlation between two like quantities.

Thus, if $O_\alpha(\underline{x}, t)$ and $O_\beta(\underline{x} + \underline{r}, t + \tau)$ are two vector operators, for example, then the autocorrelation function

$$C_{\alpha\beta}(\underline{r}, \tau) = \lim_{\substack{V\to\infty \\ T\to\infty}} \frac{1}{V} \int d\underline{x} \frac{1}{T} \int_{-T/2}^{T/2} dt\, O_\alpha(\underline{x}, t)\; O_\beta(\underline{x}+\underline{r}, t+\tau) \tag{88}$$

and the spectral density

$$S_{\alpha\beta}(\underline{k}, \omega) = \int_{-\infty}^{\infty} d\underline{r}\, d\tau\; e^{-i(\underline{k}\cdot\underline{r}+\omega\tau)} C_{\alpha\beta}(\underline{r}, \tau) \quad . \tag{89}$$

An equivalent way of writing Equation (89) for the normalized fluctuating number densities is

$$n_\alpha n_\beta S_{\alpha\beta}(\underline{k}, \omega) = \int_{-\infty}^{\infty} d\underline{r}\, dt\, e^{-i(\underline{k}\cdot\underline{r}+\omega\tau)} \langle \delta n_\alpha(\underline{x}, t)\, \delta n_\beta(\underline{x}+\underline{r}, t+\tau)\rangle \quad , \tag{90}$$

where n_α is the average number density of the α^{th} species and we assume that the electrons and ions have equal and opposite charges with $n_i = n_e = n_o$.

The spectral densities S_{ee} and S_{ii} are plotted as functions of frequency in Figure III for wavenumbers $k < k_D$ and $k > k_D$ for $\underline{B}_0 = 0$ and for distribution functions that are approximately isotropic. We note that for $k < k_D$ (i.e., the region of k-space where longitudinal oscillations are not heavily damped) the spectral density S_{ee} has a sharp resonance at $\omega \cong \omega_e$, and both S_{ee} and S_{ii} have a low frequency resonance of approximate width kV_i. Here V_e and V_i are the electron and ion thermal velocities and $k_D = (4\pi e^2 n_0/KT)^{\frac{1}{2}}$ is the Debye wavenumber. These resonances correspond to the presence of longitudinal electron and ion plasma oscillations in the fluctuation spectrum of the plasma. For $k > k_D$, where these plasma oscillations become strongly damped, the sharp resonance in S_{ee} at $\omega \cong \omega_e$ vanishes.

The inclusion of the magnetic field changes the location of this resonance, corresponding to the direction of propagation of the longitudinal wave (see Chapter I). Now we obtain a resonance in $S_{ee}(\underline{k}, \omega)$ at $\omega = \omega_0$ where $\omega_0^2 \cong \omega_e^2 + \Omega^2 \sin^2 \theta$ (see Figure I). The width of this sharp resonance is not increased in lowest order, as will be shown in the weak field expansion part (section B).

The bremsstrahlung emission from the accelerated electrons can be conveniently divided into two parts: a part from the wavenumber range $k > k_D$ and a part from $k < k_D$. The first range gives the usual collisional contribution to the emission (Dupree,[18] Dawson and Oberman,[20] Chang,[21] Oster[22]), and the second range the wave-emission due to scattering of electron plasma oscillations by ion waves and by other electron waves. Thus, we obtain our greatly enhanced (for some non-thermal distributions) emission of radiation at approximately ω_e and $2\omega_e$.

Returning to the spectral densities, we find that the expectation value of the fluctuating number densities $<\delta n_\alpha \delta n_\beta>$ can be obtained either by the formalism of Dupree[17,18] or by the test particle method of Rostoker[23] for the case when the ions are considered immobile. The formulas for the spectral densities are

$$\frac{n_0}{2} S_{ee}(\underline{k}, \omega) = \frac{\mathrm{Re}(U_e)}{|\mathscr{E}|^2} |1 - L_i|^2 + \frac{\mathrm{Re}(U_i)}{|\mathscr{E}|^2} |L_e|^2 , \tag{91}$$

and

$$\frac{n_0}{2} S_{ii}(\underline{k}, \omega) = \frac{\mathrm{Re}(U_i)}{|\mathscr{E}|^2} |1 - L_e|^2 + \frac{\mathrm{Re}(U_e)}{|\mathscr{E}|^2} |L_i|^2 , \tag{92}$$

where the arguments of all functions on the right of (91) and (92) are $(\underline{k}, i\omega)$. The longitudinal dielectric function $\mathscr{E}$ is given by

$$\mathscr{E}(\underline{k}, p) = 1 - \sum_{\alpha=e,i} L_\alpha(\underline{k}, p) \tag{93}$$

where

$$L_\alpha(\underline{k}, p) = \frac{\omega_\alpha^2}{k^2} \int_{-\infty}^{\infty} d\underline{v} \sum_{n=-\infty}^{\infty} \frac{J_n^2\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right) i \left[k_\parallel \frac{\partial f_\alpha}{\partial v_\parallel} + \frac{n\Omega_\alpha}{v_\perp} \frac{\partial f_\alpha}{\partial v_\perp}\right]}{(p + ik_\parallel v_\parallel + in\Omega_\alpha)} \tag{94}$$

and

$$U_\alpha(\underline{k}, p) = \int_{-\infty}^{\infty} d\underline{v}\, f_\alpha(\underline{v}) \sum_{n=-\infty}^{\infty} \frac{J_n^2\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right)}{(p + ik_\parallel v_\parallel + in\Omega_\alpha)} \tag{95}$$

for $\mathrm{Re}(p) > 0$ and $\Omega_\alpha = e_\alpha |\underline{B}_0| / m_\alpha C$. The distributions f_α are assumed to satisfy the requirement (from the linearized Vlasov Equation) that $(\underline{v} \times \underline{B}_0) \cdot \frac{\partial f_\alpha}{\partial \underline{v}} = 0$ (i.e., f_α is gyrotropic and is of the form $f_\alpha(v_\parallel, v_\perp^2)$). All directions are specified with respect to the uniform magnetic field $\underline{B}_0$ as indicated by the subscripts '$\|$' and '$\perp$' (refer to Figure I). In (94) and (95), J_n is a Bessel function of order n.

B. Weak Field Expansion

In this particular problem we are interested in a weak magnetic field and we make the appropriate expansion of the functions $\mathscr{E}$, $L_\alpha(\underline{k}, i\omega)$, and $U_\alpha(\underline{k}, i\omega)$, in powers of Ω_α. We may do this by writing

$$(p + ik_\parallel v_\parallel + in\Omega_\alpha)^{-1} = \int_{-\infty}^{0} d\beta \exp(p + ik_\parallel v_\parallel + in\Omega_\alpha)\beta \tag{96}$$

and using the identity

$$2\pi \sum_{n=-\infty}^{\infty} e^{in\gamma} J_n^2(z) \begin{pmatrix} 1 \\ n/z \end{pmatrix} = \int_0^{2\pi} d\phi \begin{pmatrix} 1 \\ \cos\phi \end{pmatrix} \exp - iz[\sin(\phi - \gamma) - \sin\phi] \tag{97}$$

Since $f_\alpha(\underline{v})$ is gyrotropic, we may perform the ϕ integrations in (94) and (95). Then, using (96), we interchange the summation and integration operations and use (97) to get

$$U_\alpha(\underline{k}, p) = \int_{-\infty}^{\infty} dv_\parallel \int_0^{\infty} dv_\perp v_\perp f_\alpha(\underline{v}) \int_{-\infty}^{0} d\beta \exp\{(p + ik_\parallel v_\parallel)\beta\} \cdot$$

$$\int_0^{2\pi} d\phi \exp - \frac{ik_\perp v_\perp}{\Omega_\alpha} [\sin(\phi - \Omega_\alpha \beta) - \sin\phi] , \tag{98}$$

and

$$L_\alpha(\underline{k}, p) = \frac{\omega_\alpha^2}{k^2} \int_{-\infty}^{\infty} dv_\parallel \int_0^{\infty} dv_\perp v_\perp \int_{-\infty}^{0} d\beta \exp\{(p + ik_\parallel v_\parallel)\beta\} \cdot$$

$$\int_0^{2\pi} d\phi \exp - \frac{ik_\perp v_\perp}{\Omega_\alpha}[\sin(\phi - \Omega_\alpha\beta) - \sin\phi] \left\{ k_\parallel \frac{\partial f_\alpha}{\partial v_\parallel} + k_\perp \cos\phi \frac{\partial f_\alpha}{\partial v_\perp} \right\} . \tag{99}$$

Interchanging the β and ϕ integrations in (98) and (99) we will be concerned with terms such as

$$\int_0^{2\pi} d\phi \, g(\phi) \qquad \text{and} \qquad \int_0^{2\pi} d\phi \cos\phi \, g(\phi)$$

where

$$g(\phi) = \int_{-\infty}^{0} d\beta \exp\{(p + i k_{\parallel} v_{\parallel})\beta\} \exp - \frac{i k_{\perp} v_{\perp}}{\Omega_{\alpha}} [\sin(\phi - \Omega_{\alpha}\beta) - \sin\phi] \ . \tag{100}$$

We expand the square bracket term in (100) and get

$$g(\phi) = \int_{-\infty}^{0} d\beta \exp\{(p + i k_{\parallel} v_{\parallel} + i k_{\perp} v_{\perp} \cos\phi)\beta\} \, h(\phi, \beta) \ , \tag{101}$$

where

$$h(\phi, \beta) = \exp i k_{\perp} v_{\perp} \left\{ \sin\phi \left[\frac{\Omega\beta^2}{2!} - \frac{\Omega^3\beta^4}{4!} + \ldots \right] + \cos\phi \left[- \frac{\Omega^2\beta^3}{3!} + \ldots \right] \right\} \tag{102}$$

Letting $\alpha = p + i k_{\parallel} v_{\parallel} + i k_{\perp} v_{\perp} \cos\phi$ for convenience and integrating $g(\phi)$ successively by parts, we have

$$g(\phi) = \frac{1}{\alpha} - \frac{1}{\alpha^2} \left(\frac{\partial h}{\partial\beta}\right)_{\beta=0} + \frac{1}{\alpha^3} \left(\frac{\partial^2 h}{\partial\beta^2}\right)_{\beta=0} - \frac{1}{\alpha^4} \left(\frac{\partial^3 h}{\partial\beta^3}\right)_{\beta=0} + \ldots \tag{103}$$

Evaluating this series, we find

$$g(\phi) = \frac{1}{\alpha} + \Omega_{\alpha} \left[\frac{i k_{\perp} v_{\perp} \sin\phi}{\alpha^3} \right] + \Omega_{\alpha}^2 \left[\frac{1}{\alpha^4} i k_{\perp} v_{\perp} \cos\phi - \frac{3}{\alpha^5} k^2 v^2 \sin^2\phi \right]$$

$$+ \text{(higher order terms in } \Omega_{\alpha}) \tag{104}$$

Note that the integrals $\int_0^{2\pi} d\phi \, \frac{\sin\phi}{\alpha^3} (\cos\phi)^n$ vanish since $\sin\phi$ is odd around $\phi = \pi$ and α^3 and $(\cos\phi)^n$ are even $(n = 0, 1)$. Thus our first correction to $g(\phi)$ is of order Ω_α^2.

Equation (98) now becomes

$$U_\alpha(\underline{k}, p) = \int_{-\infty}^{\infty} dv_\parallel \int_0^{\infty} dv_\perp v_\perp f_\alpha(\underline{v}) \int_0^{2\pi} d\phi \left\{ \frac{1}{p + i k_\parallel v_\parallel + i k_\perp v_\perp \cos\phi} \right.$$

$$\left. + \Omega_\alpha^2 \left[\frac{i k v \cos\phi}{(p + i k_\parallel v_\parallel + i k v \cos\phi)^4} - \frac{3 k^2 v^2 \sin^2\phi}{(p + i k_\parallel v_\parallel + i k v \cos\phi)^5} \right] + \ldots \right\}, \tag{105}$$

which can be written in Cartesian coordinates as

$$U_\alpha(\underline{k}, p) = \int_{-\infty}^{\infty} d\underline{v}\, f_\alpha(\underline{v}) \left\{ \frac{1}{p + i \underline{k} \cdot \underline{v}} + \Omega_\alpha^2 \left[\frac{i k_x v_x}{(p + i \underline{k} \cdot \underline{v})^4} - \frac{3 k_x^2 v_y^2}{(p + i \underline{k} \cdot \underline{v})^5} \right] \right\} \tag{106}$$

Similarly,

$$L_\alpha(\underline{k}, p) = \frac{\omega_\alpha^2}{k^2} \int_{-\infty} d\underline{v}\, \underline{k} \cdot \frac{\partial f_\alpha}{\partial \underline{v}}(\underline{v}) \left\{ \frac{1}{p + i \underline{k} \cdot \underline{v}} + \Omega_\alpha^2 \left[\frac{i k_x v_x}{(p + i \underline{k} \cdot \underline{v})^4} - \right. \right.$$

$$\left. \left. - \frac{3 k_x^2 v_y^2}{(p + i \underline{k} \cdot \underline{v})^5} \right] \right\} . \tag{107}$$

To the lowest order of significance for our problem we find

$$\mathrm{Im}[\,\mathscr{E}(\underline{k}, i\omega)] \cong \sum_\alpha \frac{\pi \omega_\alpha^2}{k^2} F'_\alpha(-\underline{k}, \frac{\omega}{k}) \quad , \tag{108}$$

$$\mathrm{Re}[U_\alpha(\underline{k}, i\omega)] \cong \frac{\pi}{k} F_\alpha(-\underline{k}, \frac{\omega}{k}) \quad , \tag{109}$$

and

and

$$\mathrm{Re}[\mathscr{E}(\underline{k}, i\omega)] \cong 1 - \frac{\omega_e^4}{\omega^4}[\omega^2 + 3(k_\perp^2 V_\perp^2 + k_\parallel^2 V_\parallel^2 + k_\parallel^2 U^2) + \frac{\Omega_e^2 k_\perp^2}{k^2} - 2\omega k_\parallel U] \quad . \tag{110}$$

We have included only the electron contribution in (110) since $m_i >> m_e$. As usual, the reduced distributions F_α are defined by

$$F_\alpha(\underline{k}, u) = \int_{-\infty}^{\infty} d\underline{v}\, f_\alpha(\underline{v})\, \delta(u - \frac{\underline{k}\cdot\underline{v}}{k}) \quad , \tag{111}$$

and

$$\int_{-\infty}^{\infty} d\underline{v}\, f_e(\underline{v}) \begin{pmatrix} v_\parallel \\ v_\perp^2 \\ (v_\parallel - U)^2 \end{pmatrix} \equiv \begin{pmatrix} U \\ 2V_\perp^2 \\ V_\parallel^2 \end{pmatrix} \quad . \tag{112}$$

Here U is an average drift velocity along $\underline{B}_0$ for the electrons and $V_\perp$ is the thermal velocity for each of the two directions perpendicular to $\underline{B}_0$.

Since we are only interested in the wave-emission contribution to the bremsstrahlung in the neighborhood of ω_e and $2\omega_e$, we need only approximate expressions for the spectral densities S_{ee}, S_{ii} for $k < k_D$ and at $\omega \cong \omega_e$ and 0 respectively. We make the same approximations as those made by Tidman and Dupree[5] and for our purposes we may write

$$\frac{n_0}{2} S_{ee}(\underline{k}, |\omega| \cong \omega_e) \cong \frac{\mathrm{Re}(U_e)}{|\mathscr{E}|^2}$$

$$\cong \mathrm{Re}(U_e)\left\{\left[1 - \frac{\omega_e^2}{\omega^2} - \frac{\omega_e^2 \Omega^2 \sin^2\theta}{\omega^2} - \frac{3\omega_e^2}{\omega^4}(k^2V^2 + k_\parallel^2 V_\parallel^2 + k_\parallel^2 U^2) + \frac{2k_\parallel U\omega_e^2}{\omega^3}\right]^2 + [\mathrm{Im}(\mathscr{E})]^2\right\}^{-1} \tag{113}$$

As in Ref. 5, we have neglected the finite line width of the Landau damping decrement in (113) ($\gamma_L = (\pi\omega_e^3/2k^2)F_e'(\omega/k) = (\omega_e/2)\,\mathrm{Im}(\mathscr{E})$). The resonance denominator in (113) can, therefore, be represented by δ-functions for purposes of integration. Note that while we desire γ_L to be very small we do not want it to become exactly zero, as this would indicate the onset of instability. We see, also, from the structure of (113) and the above approximations, that the influence of the magnetic field does indeed first manifest itself in the Re $[\mathscr{E}]$, thus justifying (108), (109) and (110).

We obtain

$$S_{ee}(\underline{k}, |\omega| \cong \omega_e) \cong \frac{\pi k \omega_0}{n_0 \omega_e^2} \frac{F_e(-\underline{k}, \omega/k)}{|F_e'(-\underline{k}, \omega/k)|} \cdot [\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] \quad , \tag{114}$$

where

$$\omega_0 = \{\omega_e^2 + \Omega^2 \sin^2\theta + 3(k^2V^2 + k_\parallel^2 V_\parallel^2 + k_\parallel^2 U^2) - 2\omega_e k_\parallel U\}^{\frac{1}{2}} \quad . \tag{115}$$

In Equation (114), therefore, $\Omega = |\Omega_e|$ and we have used the inequalities $\Omega^2 \ll \omega_e^2$, $|\mathrm{Im}(\mathscr{E})| \ll 1$.

Similarly both the ion and electron spectral densities at low frequencies can be obtained by taking the limit $m_i \to \infty$ and approximating with a δ-function at $\omega = 0$. Thus, we have $S_{ii}(\underline{k}, \omega \cong 0) \cong \frac{\pi}{n_0}\delta(\omega) \cong S_{ee}(\underline{k}, \omega \cong 0)$.

C. Wave-Emission Formulas and Small Wavenumber Magnetic Field Effects

Equations (86) and (87) give the rate of excitation of a radiation mode of wavenumber $\underline{K}$ and polarization $\underline{\epsilon}_0$. It is convenient to average the polarization vectors $\underline{\epsilon}_0$ in the plane perpendicular to $\underline{K}$ for any given $\underline{K}$. We choose, therefore, a coordinate system with $\underline{K}$ along the polar (z) axis and $\underline{\epsilon}_0$ in the xy plane. Then if $\underline{\epsilon}_0$ makes an azimuthal angle ϕ about $\underline{K}$ measured from the plane defined by $\underline{k}$ and $\underline{K}$ we have

$$|\underline{k}\cdot\underline{\epsilon}_0|^2 = |\underline{\epsilon}_0|^2\, k^2 \cos^2\phi \left[1 - \frac{(\underline{k}\cdot\underline{K})^2}{k^2K^2}\right] , \tag{116}$$

and

$$\left\langle|\underline{k}\cdot\underline{\epsilon}_0|^2\right\rangle = \frac{1}{2\pi}\int_0^{\pi} d\phi\, |\underline{k}\cdot\underline{\epsilon}_0|^2 = \pi k^2 \left[1 - \frac{(\underline{k}\cdot\underline{K})^2}{k^2\,K^2}\right] , \tag{117}$$

since $|\underline{\epsilon}_0|^2 = 2\pi$. Next we define the emitted intensity

$$\frac{d^2 I}{d\omega\, d\sigma} = \sum \left(\frac{dU}{dt}\right) \frac{d^2 n}{d\omega\, d\sigma} \text{ ergs (sec. cps. sterad)}^{-1} , \tag{118}$$

where the density of states per unit solid angle in the direction $\underline{K}$ for the radiation field is for each polarization

$$\frac{d^2 n}{d\omega\, d\sigma} = \frac{K^2}{8\pi^3}\frac{dK}{d\omega} \quad , \tag{119}$$

and the summation in (118) is over the two transverse polarizations.

Using (118), we substitute the reduced forms (114) and (117) into (86) and (87) and integrate over ω' (the frequency of the longitudinal waves) to obtain the radiated intensities near ω_e and $2\omega_e$. Thus the wave-emission becomes

$$\left(\frac{d^2 I_1}{d\sigma\, d\omega}\right)^{\text{wave}} \cong \sum \frac{e^2 \quad \omega_e^2 K^2}{(2\pi)^4 \omega^2 8}\frac{dK}{d\omega}\int_{k<k_D}\frac{d\underline{k}}{k}\left[1 - \frac{(\underline{k}\cdot\underline{K})^2}{k^2 K^2}\right] \cdot$$

$$\left[\frac{F_e(\underline{k}, \frac{\omega_0}{k})}{|F_e'(\underline{k}, \frac{\omega_0}{k})|} + \frac{F_e(-\underline{k}, \frac{\omega_0}{k})}{|F_e'(-\underline{k}, \frac{\omega_0}{k})|}\right] \delta(\omega - \omega_0)\,\omega_0 \quad , \tag{120}$$

and

$$\left(\frac{d^2 I_2}{d\sigma\, d\omega}\right)^{\text{wave}} \cong \sum \frac{e^2 \; K^2}{(2\pi)^4 4\omega^2}\frac{dK}{d\omega}\int_{k<k_D} d\underline{k}\,\frac{(\underline{k}\cdot\underline{K})^2}{k^4}\left[1 - \frac{(\underline{k}\cdot\underline{K})^2}{k^2 K^2}\right]$$

$$\left\{\frac{F_e(-\underline{k}, \frac{\omega_0}{k})\,F_e(\underline{k}, \frac{\omega-\omega_0}{k})}{|F_e'(-\underline{k}, \frac{\omega_0}{k})\,F_e'(\underline{k}, \frac{\omega-\omega_0}{k})|} + \frac{F_e(\underline{k}, \frac{\omega_0}{k})\,F_e(-\underline{k}, \frac{\omega-\omega_0}{k})}{|F_e'(\underline{k}, \frac{\omega_0}{k})\,F_e(-\underline{k}, \frac{\omega-\omega_0}{k})|}\right\} \delta(\omega - \omega_0)\,\omega_0^2 \, , \tag{121}$$

where we have used the symmetry of F_e (see (111)) with respect to the sign of its argument.

In (120) and (121), we can also utilize the δ-functions implicitly to simplify these expressions. That is, in (120) $\omega = \omega_0 \cong \omega_e$, and in (121), $\omega = 2\omega_0 \cong 2\omega_e$.

Hence

$$\left(\frac{d^2 I_1}{d\sigma\, d\omega}\right)^{\text{wave}} \cong \sum \frac{e^2\, \omega_e K^2}{(2\pi)^4\, 8} \frac{dK}{d\omega} \int_{k<k_D} \frac{d\underline{k}}{k}\left[1 - \frac{(\underline{k}\cdot\underline{K})^2}{k^2 K^2}\right] \cdot$$

$$\left[\frac{F_e(\underline{k}, \frac{\omega_0}{k})}{|F_e{}'(\underline{k}, \frac{\omega_0}{k})|} + \frac{F_e(-\underline{k}, \frac{\omega_0}{k})}{|F_e{}'(-\underline{k}, \frac{\omega_0}{k})|}\right] \delta(\omega - \omega_0) , \qquad (122)$$

and

$$\left(\frac{d^2 I_2}{d\sigma\, d\omega}\right)^{\text{wave}} \cong \sum \frac{e^2 K^2}{(2\pi)^4\, 8} \frac{dK}{d\omega} \int_{k<k_D} d\underline{k} \frac{(\underline{k}\cdot\underline{K})^2}{k^4}\left[1 - \frac{(\underline{k}\cdot\underline{K})^2}{k^2 K^2}\right] \cdot$$

$$\frac{F_e(\underline{k}, \frac{\omega_0}{k}) F_e(-\underline{k}, \frac{\omega_0}{k})}{|F_e{}'(\underline{k}, \frac{\omega_0}{k}) F_e{}'(-\underline{k}, \frac{\omega_0}{k})|} \delta(\omega - 2\omega_0) \qquad (123)$$

We now have (122) and (123) to represent the emission obtained when ion or electron plasma oscillations collide nearly head-on with other electron plasma oscillations (refer to Figure II).

We shall consider only those situations in which the major contribution to (122) and (123) occurs for small longitudinal wavenumber k. In this range, therefore, we may make an important simplification in the dispersion relation (115) and neglect thermal effects and drifts. Thus the magnetic field effects dominate in this region, although the field is still 'weak' in the sense $\Omega^2 << \omega_e^2$. The magnetic field manifests itself, therefore, in the modified dispersion relation

$$\omega_0 \cong (\omega_e^2 + \Omega^2 \sin^2 \theta)^{\frac{1}{2}} \quad . \tag{124}$$

Similarly, the dispersion relation for electromagnetic waves is

$$\omega^2 \cong \omega_e^2 + c^2 K^2 \pm \left(\frac{\omega_e^2 \Omega}{\omega}\right) \cos \psi \quad , \tag{125}$$

where in (124) and (125) the angles θ and ψ are those between $\underline{k}$ and $\underline{B}_0$ and between $\underline{K}$ and $\underline{B}_0$ respectively (see Figure II). The $\pm$ signs in (125) denote a small splitting for the two polarizations, and it should be noted that this correction term in (125) for $\Omega^2 << \omega_e^2$ is valid only for ψ such that $\cos \psi \gtrsim \frac{\Omega}{2\omega_e}$.

Next, we consider the density of states factor $(K^2/8\pi^3)(dK/d\omega)$ in (122) and (123). From (125) it follows that

$$\frac{d^2 n}{d\omega \, d\sigma} = \frac{K^2}{8\pi^3} \frac{dK}{d\omega} = \frac{\left(\omega \pm \frac{\omega_e \Omega \cos \psi}{2\omega^2}\right)}{8\pi^3 c^3} \quad (\omega^2 - \omega_e^2 \mp \frac{\omega_e^2 \Omega}{\omega} \cos \psi)^{\frac{1}{2}} \, . \tag{126}$$

As a consequence of the δ-function $\delta(\omega - 2\omega_0)$ and $\omega_0 \cong \omega_e$ in (123) we can write

$$\frac{K^2}{8\pi^3} \frac{dK}{d\omega} \cong \frac{\sqrt{3}\, \omega_e^2}{4\pi^3 c^3} \quad . \tag{127}$$

Similarly in (122) we use the δ-function $\delta(\omega - \omega_0)$ and $\omega_0 \cong \omega_e$ to obtain

$$\frac{K^2}{8\pi^3}\frac{dK}{d\omega} \cong \frac{\omega_e}{8\pi^3 c^3}\left|\omega_e \Omega \cos\psi\right|^{\frac{1}{2}} \quad , \tag{128}$$

since for nearly all θ and ψ we have $\Omega^2 \sin^2\theta << |\omega_e \Omega \cos\psi|$. From the square root in (126) we note that only one polarization propagates for the fundamental. The other polarization must, therefore, be excluded from the summation in (121). However, both polarizations at $\omega \cong 2\omega_e$ propagate and have a density of states given by (127). Thus we may write (122) and (123) as

$$\left(\frac{d^2 I_1}{d\sigma\, d\omega}\right)^{\text{wave}} \cong \frac{e^2 \omega_e^2 |\omega_e \Omega \cos\psi|^{\frac{1}{2}}}{8c^3(2\pi)^4} \int_{k<k_D} \frac{d\underline{k}}{k}\left[1 - \frac{(\underline{k}\cdot\underline{K})}{k^2 K^2}\right] \cdot$$

$$\cdot \left[\frac{F_e(\underline{k}, \frac{\omega_0}{k})}{|F_e'(\underline{k}, \frac{\omega_0}{k})|} + \frac{F_e(-\underline{k}, \frac{\omega_0}{k})}{|F_e'(-\underline{k}, \frac{\omega_0}{k})|}\right] \delta(\omega - \omega_0) \quad , \tag{129}$$

and

$$\left(\frac{d^2 I_2}{d\sigma\, d\omega}\right)^{\text{wave}} \cong \frac{e^2 \sqrt{3}\, \omega_e^2}{2c^3(2\pi)^4} \int_{k<k_D} d\underline{k}\, \frac{(\underline{k}\cdot\underline{K})^2}{k^4}\left[1 - \frac{(\underline{k}\cdot\underline{K})^2}{k^2 K^2}\right]$$

$$\cdot \frac{F_e(\underline{k}, \frac{\omega_0}{k}) F_e(-\underline{k}, \frac{\omega_0}{k})}{|F_e'(\underline{k}, \frac{\omega_0}{k}) F_e'(-\underline{k}, \frac{\omega_0}{k})|} \delta(\omega - 2\omega_0) \, , \tag{130}$$

where $\omega_0^2 \cong \omega_e^2 + \Omega^2 \sin^2\theta$.

We see that the angular dependence of ω_0 introduced via the magnetic field now operates through the δ-functions in (129) and (130) to relate the emitted frequency ω to an equivalent angle. Thus, the angular dependence of the reduced distributions will directly affect the frequency dependence of the emitted radiation.

D. Angle Integrations and Some General Considerations

In order to illustrate the effects of the magnetic field more clearly and discuss the line shape of the emitted radiation, it is convenient to carry out the angle integrations in (129) and (130). Thus (129) becomes

$$\left(\frac{d^2 I_1}{d\sigma\, d\omega}\right)^{\text{wave}} \cong \frac{e^2 \omega_e^2 \left|\omega_e \Omega \cos\psi\right|^{\frac{1}{2}} \omega}{4c^3 (2\pi)^3 \Omega^2 C_1} \left[1 - \tfrac{1}{2} S_1^2 \sin^2\psi - C_1^2 \cos^2\psi\right] \cdot$$

$$\cdot\, W_1(S_1, C_1; \omega) \quad , \qquad (131)$$

where

$$W_1(S_1, C_1; \omega) = \int_0^{k_D} k dk \left[\frac{F_e(\underline{k}, \frac{\omega}{k})}{\left|F_e'(\underline{k}, \frac{\omega}{k})\right|} + \frac{F_e(-\underline{k}, \frac{\omega}{k})}{\left|F_e'(-\underline{k}, \frac{\omega}{k})\right|}\right] \qquad (132)$$

$$\begin{pmatrix} \cos\theta \rightarrow C_1 \\ \sin\theta \rightarrow S_1 \end{pmatrix}$$

with $\cos\theta$ and $\sin\theta$ in the F_e functions replaced by C_1 and S_1 respectively. We have

$$\begin{aligned} S_1 &= \frac{1}{\Omega}(\omega^2 - \omega_e^2)^{\frac{1}{2}} \\ C_1 &= \frac{1}{\Omega}(\Omega^2 - \omega^2 + \omega_e^2)^{\frac{1}{2}} \end{aligned} \qquad (133)$$

with W_1 non-zero only in the frequency range $\omega_e < \omega < (\omega_e^2 + \Omega)^{\frac{1}{2}}$.

The angles θ and ψ have the same meaning as before (see Figure II).

Similarly (130) becomes

$$\left(\frac{d^2 I_2}{d\sigma\, d\omega}\right)^{\text{wave}} \cong \frac{e^2 \sqrt{3}\, \omega_e^2 K^2 \omega}{4c^3 (2\pi)^3 \Omega^2 C_2} W_2(S_2\,,\, C_2\,;\, \omega) \cdot$$

$$\cdot \{C_2^2 \cos^2\psi + \frac{1}{2} S_2^2 \sin^2\psi - C_2^4 \cos^4\psi - \frac{3}{8} S_2^4 \sin^4\psi - 3S_2^2 C_2^2 \cos^2\psi \sin^2\psi\} \,, \tag{134}$$

where

$$W_2(S_2\,,\, C_2;\, \omega) = \int_0^{k_D} dk \left\{ \frac{F_e(\underline{k}, \frac{\omega}{2k}) F_e'(-\underline{k}, \frac{\omega}{2k})}{|F_e'(\underline{k}, \frac{\omega}{2k}) F_e'(-\underline{k}, \frac{\omega}{2k})|} \right\} \,, \tag{135}$$

$$\begin{pmatrix} \cos\theta \rightarrow C_2 \\ \sin\theta \rightarrow S_2 \end{pmatrix}$$

and

$$\begin{aligned} S_2 &= \frac{1}{2\Omega} (\omega^2 - 4\omega_e^2)^{\frac{1}{2}} \\ C_2 &= \frac{1}{2\Omega} (4\Omega^2 - \omega^2 + 4\omega_e^2)^{\frac{1}{2}} \end{aligned} \tag{136}$$

Here W_2 is non-zero only in the frequency range $2\omega_e < \omega < 2(\omega_e^2 + \Omega^2)^{\frac{1}{2}}$.

Now consider the weighting functions W_1 and W_2. For <u>isotropic</u> distribution functions, F_e is not a function of the direction of $\underline{k}$; W_1 and W_2, therefore, will be independent of S_1 and C_1, and S_2 and C_2 respectively. Thus $W_{1,2}$ will be almost constant over the allowed frequency range, since they vary weakly with ω. This can be seen from consideration of (132) and (135) with some appropriate isotropic F_e. If we use Maxwellians, for example, as our original distribution (e.g., a thermal part plus an energetic part), we will find to a good approximation that $W_1 \propto \omega^{-1}$ and $W_2 \propto \omega^{-2}$. Note that both (131) and (134) already have a factor ω in the numerator. The divergence at the upper edge of both of the differential intensities (131) and (134) originates in the factors C_1^{-1} and C_2^{-1} respectively. The weighting function ωW_1 and the differential intensity for the fundamental are illustrated schematically in Figure IV. The terms C_1^{-1} and C_2^{-1} derive from a solid angle effect in that there are more longitudinal $\underline{k}$ vectors propagating at large angles to $\underline{B}_0$ than at small angles to $\underline{B}_0$. They contribute a finite but small amount to the observed intensity in any finite frequency band $\Delta\omega$ around $(\omega_e^2 + \Omega^2)^{\frac{1}{2}}$ and $2(\omega_e^2 + \Omega^2)^{\frac{1}{2}}$ as can be seen by taking the integral over ω over the factors $(\Omega^2 - \omega^2 + \omega_e^2)^{-\frac{1}{2}}$ and $(4\Omega^2 - \omega^2 + 4\omega_e^2)^{-\frac{1}{2}}$ in (131) and (134) respectively.

Thus we see that isotropic distributions in the presence of a magnetic field lead only to broadening of the emission lines through almost uniform enhancement of the emission over the allowed range of ω. For a clearly observable splitting, we want the intensity at the outer edges of the two lines to be at least one order of magnitude greater than at the centers of the lines. This will be true in general if

$$\begin{aligned} W_{1,2}(0,\ 1;\ \omega_L) &>> W_{1,2}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\ ;\ \omega_M\right) \\ W_{1,2}(1,\ 0;\ \omega_U) &> W_{1,2}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\ ;\ \omega_M\right) \end{aligned} \tag{137}$$

where ω_L $(=\omega_{Lower}) = \omega_e$ and $2\omega_e$ for W_1 and W_2 respectively, ω_U $(=\omega_{Upper}) = (\omega_e^2+\Omega^2)^{\frac{1}{2}}$ and $2(\omega_e^2+\Omega^2)^{2}$ respectively, and ω_M $(=\omega_{Midrange}) \cong \omega_e + \frac{\Omega^2}{4\omega_e}$ and $2\omega_e + \frac{\Omega^2}{2\omega_e}$ respectively, for the range of $\underline{k}$ vectors in which electron plasma oscillations are enhanced. Note that the observed intensity at the upper edge of the lines is helped by the solid angle effect, thus reducing the magnitude requirements on $W_{1,2}(1,\ 0;\ \omega_U)$.

Clearly, we may achieve condition (137) only if we use distributions which are anisotropic in θ . In the following calculations we shall use a tenuous flux of superthermal electrons coexisting with a thermal or Maxwellian background. These energetic electrons provide both the driving mechanism for the enhanced emission and, hopefully, the requisite anisotropy. Thus, if we have an anisotropic superthermal distribution which concentrates the longitudinal $\underline{k}$ vectors in the two directions parallel and perpendicular to $\underline{B}_0$, we may obtain an observable splitting of the emission lines through the preceding formalism.

We shall consider distributions of the general type that have $T_\perp >> T_\parallel$ and are also drifting through the background plasma along $\underline{B}_0$ (all directions are with respect to $\underline{B}_0$). We do not know, of course, the details of the electron distribution function at the source of a Type II radio burst. However, the above distributions seem physically plausible and can explain the observed spectral features of the radiation.

E. Results for a Drifting Maxwellian

We now consider specific distributions which fall into the general category discussed above. Thus we will first assume a distribution of the following form:

$$f_e(\underline{v}) = \frac{\beta}{(2\pi)^{\frac{3}{2}} V_e^3} \exp\left(-\frac{v^2}{2V_e^2}\right) + \frac{(1-\beta)}{(2\pi)^{\frac{3}{2}} V_{\parallel} V_{\perp}^2} \cdot$$

$$\exp\left[-\frac{v_{\perp}^2}{2V_{\perp}^2} - \frac{(v_{\parallel} - U)^2}{2V_{\parallel}^2}\right] \tag{138}$$

where $V_{\perp}^2 >> V_e^2$, $V_{\perp}^2 >> V_{\parallel}^2$, and we also choose $(1-\beta) << \beta \cong 1$. Here, $\underline{v}_{\parallel}$ and $\underline{v}_{\perp}$ are the electron velocity components parallel and perpendicular to $\underline{B}_0$ respectively. Thus the superthermal electrons form a sombrero-like distribution in velocity space drifting through the thermal background of electrons with a drift velocity U of such a magnitude that the distribution is stable.

We insert (138) into (111) and obtain the reduced distribution function F_e used in W_1 and W_2 (with the appropriate arguments),

$$F_e(\underline{k}, \frac{\omega}{k}) = \frac{\beta}{(2\pi)^{\frac{1}{2}} V_e} \exp -\left[\frac{\omega^2}{2k^2 V_e^2}\right]$$

$$+ \frac{k(1-\beta)}{(2\pi)^{\frac{1}{2}} (k_{\perp}^2 V_{\perp}^2 + k_{\parallel}^2 V_{\parallel}^2)^{\frac{1}{2}}} \exp\left[-\frac{(\omega - k_{\parallel} U)^2}{2(k_{\perp}^2 V_{\perp}^2 + k_{\parallel}^2 V_{\parallel}^2)}\right]$$

$$= F_T + F_E \tag{139}$$

and

$$F'_e(\underset{-}{k}, \frac{\omega}{k}) = \frac{-\beta\omega}{(2\pi)^{\frac{1}{2}} k V_e^3} \exp\left[-\frac{\omega^2}{2k^2 V_e^2}\right] -$$

$$\frac{k^2(1-\beta)(\omega - k_\parallel U)}{(2\pi)^{\frac{1}{2}}(k_\perp^2 V_\perp^2 + k_\parallel^2 V_\parallel^2)^{\frac{3}{2}}} \exp\left[-\frac{(\omega - k_\parallel U)^2}{2(k_\perp^2 V_\perp^2 + k_\parallel^2 V_\parallel^2)}\right] \qquad (140)$$

When these functions are used in $W_{1,2}$, $k_\perp = k \sin\theta$ becomes $k S_{1,2}$ and $k_\parallel = k \cos\theta$ becomes $k C_{1,2}$ respectively.

We now want to obtain approximate expressions for the W functions for both the fundamental and second harmonic to determine whether or not observable splitting is generated. In the following discussion, however, our general remarks will apply equally well to either line. We need, therefore, only consider the fundamental in detail to illustrate our points.

For the fundamental, therefore, consider first $\theta = \pi/2$. Then our expressions simplify to

$$F_e(\underset{-}{k}, \frac{\omega}{k}) = F_e(-\underset{-}{k}, \frac{\omega}{k}) = F_T(\theta = \frac{\pi}{2}) + F_E(\frac{\pi}{2}) \quad . \qquad (141)$$

In this case,

$$F_T = F_{\text{Thermal}} = \frac{\beta}{(2\pi)^{\frac{1}{2}} V_e} \exp\left[-\frac{\omega^2}{2k^2 V_e^2}\right] , \qquad (142)$$

and

$$F_E = F_{\text{Energetic}} = \frac{(1-\beta)}{(2\pi)^{\frac{1}{2}} V_\perp} \exp\left[-\frac{\omega^2}{2k^2 V_\perp^2}\right] , \qquad (143)$$

with $F'_T = -\frac{\omega}{kV_e^2} F_T$ and $F'_E = -\frac{\omega}{kV_\perp^2} F_E$.

Thus (132) can be written (for $\theta = \pi/2$)

$$W_1(1, 0, \omega_U) = \int_0^{k_D} 2k\,dk \, \frac{(F_T + F_E)}{|F_T' + F_E'|} \tag{144}$$

which in turn can be written as

$$W_1(1, 0; \omega_U) = \int_0^{k_D} 2k\,dk \, \frac{kV_\perp^2}{\omega} G(k) \tag{145}$$

where

$$G(k) = \frac{1 + \dfrac{F_E}{F_T}}{\dfrac{V_\perp^2}{V_e^2} + \dfrac{F_E}{F_T}} \tag{146}$$

and

$$\frac{F_E}{F_T} = \frac{(1-\beta)}{\beta} \frac{V_e}{V_\perp} \exp \frac{\omega^2}{2k^2 V_e^2} \left(1 - \frac{V_e^2}{V_\perp^2}\right) . \tag{147}$$

Thus, we have to a good approximation

$$\frac{F_E}{F_T} \cong (1-\beta) \frac{V_e}{V_\perp} \exp \frac{k_D^2}{2k^2} \tag{148}$$

Examination of (146) and (148) for typical parameters such as $(1-\beta) \sim 0(10^{-2})$ and $V_\perp^2/V_e^2 \sim 0(10^3)$, shows that $G(k)$ may be represented schematically as shown in Figure V. Note, k_1 is chosen so that $\dfrac{F_E}{F_T} = \dfrac{V_\perp^2}{V_e^2}$, hence $G(k_1) = \dfrac{1}{2}$; k_2 is chosen to yield $\dfrac{F_E}{F_T} = 1$, hence $G(k_2) = 2V_e^2/V_\perp^2$, with $G(k_D) = V_e^2/V_\perp^2$. Typical values for k_1 and k_2 (for the above values of $(1-\beta)$ and $V_\perp^2/V_e^2$) are $\sim k_D/5.5$

and $\sim k_D/4$ respectively, where k_1 and k_2 are given by

$$k_1 \cong \frac{k_D}{\sqrt{2}} \left\{ \log \frac{V_\perp^3}{V_e^3 (1 - \beta)} \right\}^{-\frac{1}{2}} \tag{149}$$

and

$$k_2 \cong \frac{k_D}{\sqrt{2}} \left\{ \log \frac{V_\perp}{V_e (1 - \beta)} \right\}^{-\frac{1}{2}} \quad . \tag{150}$$

Unless otherwise indicated, all logarithms are to the base e .

It seems reasonable, therefore, to divide the k integration into two parts. Thus we write

$$W_1 (1, 0, \omega_U) \cong \int_0^{k_1} 2k\,dk \, \frac{kV_\perp^2}{\omega} \, G_1(k) + \int_{k_1}^{k_D} 2k\,dk \, \frac{kV_\perp^2}{\omega} G_2(k) \tag{151}$$

where

$$G_1(k) = 1 \tag{152}$$

and

$$G_2(k) = \frac{V_e^2}{V_\perp^2} \, \frac{1}{k_2 - k_D} [k - 2k_D + k_2] \quad . \tag{153}$$

We have, therefore,

$$W_1(1, 0, \omega_U) \cong \frac{2}{3} \frac{k_1^3 V_\perp^2}{\omega} + \frac{5}{6} \frac{V_e^2}{\omega} (k_D^3 - k_1^3) \quad , \tag{154}$$

where we have taken

$$G_2 \cong 2 \frac{V_e^2}{V_\perp^2} \left(1 - \frac{k}{2k_D} \right) \quad . \tag{155}$$

We can assume

$$W_1(1, 0, \omega_U) \cong \frac{2}{3} \frac{k_1^3 V_\perp^2}{\omega} \tag{156}$$

providing $k_i^3 V_\perp^2 > k_D^3 V_e^2$. This condition, which can be restated as

$V_\perp^2 / V_e^2 > 2\sqrt{2} \left\{ \log \frac{V_\perp^3}{V_e^3(1-\beta)} \right\}^{\frac{3}{2}}$, is easily satisfied.

In a similar fashion, we have

$$W_1\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \omega_M\right) = \int_0^{k_D} k\,dk \, \frac{kV_\perp^2}{\omega} \{G^+(k) + G^-(k)\}, \tag{157}$$

where

$$G^\pm(k) \cong \frac{1 + \frac{F_E^\pm}{F_T}}{\left| \frac{V_\perp^2}{V_e^2} + 2\left(1 \pm \frac{kU}{\sqrt{2}\,\omega}\right) \frac{F_E^\pm}{F_T} \right|}, \tag{158}$$

and

$$\frac{F_E^\pm}{F_T} \cong \frac{(1-\beta)}{\beta} \frac{V_e}{V_\perp} \sqrt{2} \exp\left\{ \frac{\omega^2}{2k^2V_e^2} - \frac{\left(\omega \mp \frac{kU}{\sqrt{2}}\right)^2}{k^2 V_\perp^2} \right\}. \tag{159}$$

To maintain a stable distribution, we shall require that $k_D U/\sqrt{2}\,\omega_e$ is < 1 in this calculation. Thus

$$\frac{F_E^\pm}{F_T} \cong (1-\beta)\sqrt{2}\,\frac{V_e}{V_\perp} \exp \frac{\omega^2}{2k^2V_e^2} \quad . \tag{160}$$

Over the k range that contributes the dominant part of the integration $G^+(k) \sim G^-(k)$, and we find

$$W_1\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \omega_M\right) \cong \frac{1}{2} W_1(1, 0, \omega_U) \quad , \tag{161}$$

which meets the general requirements stated in the previous section.

We see that the $V_\perp^2$ factor in (156) can lead to a considerable enhancement in emission as compared to thermal equilibrium.

Now we consider $\theta = 0$ (i.e., $W_1(0, 1, \omega_L)$)

$$W_1(0, 1, \omega_L) = \int_0^{k_D} k\,dk \, \frac{kV_\parallel^2}{\omega} \{G^+(k) + G^-(k)\} \tag{162}$$

where now

$$G^\pm(k) = \frac{1 + \frac{F_E^\pm}{F_T}}{\left| \frac{V_\parallel^2}{V_e^2} + \left(1 \mp \frac{kU}{\omega}\right) \frac{F_E^\pm}{F_T} \right|} \tag{163}$$

and

$$\frac{F_E^\pm}{F_T} = \frac{1-\beta}{\beta} \frac{V_e}{V_\parallel} \exp\left\{ \frac{\omega^2}{2k^2V_e^2} - \frac{(\omega \mp kU)^2}{2k^2V_\parallel^2} \right\} . \tag{164}$$

Recall $V_\parallel^2 \ll V_\perp^2$. In fact, $V_\parallel^2$ may be of the same order of magnitude as V_e^2 . Clearly, the only contribution which is comparable to (or larger than) $W_1(1, 0, \omega_U)$ and $W_1\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \omega_M\right)$ must come from $G^+(k)$ when the denominator of (163) has the opportunity to become very small. Thus,

$$W_1(0, 1, \omega_L) \cong \int_0^{k_D} k\,dk \, \frac{kV_\parallel^2}{\omega} G^+(k) \tag{165}$$

At this point, it is more convenient to rewrite $W_1(0, 1; \omega_L)$ in its original form as indicated by (132), namely,

$$W_1(0, 1; \omega_L) \cong \int_0^{k_D} k\,dk \; \frac{F_e(\underline{k}, \frac{\omega}{k})}{\left| F_e'(\underline{k}, \frac{\omega}{k}) \right|} \quad . \tag{166}$$

We take $k F_e(\underline{k}, \frac{\omega}{k})$ to be a very slowly varying function in the vicinity of the minimum of $F_e'(\underline{k}, \frac{\omega}{k})$ and Taylor expand F_e' about its minimum which occurs at $k = k_s$. At this wavenumber, for which $\frac{\partial F_e'}{\partial k} = 0$, the distribution is verging on instability. That is, k_s is the "most nearly unstable" wavenumber. We do not actually allow F_e' to vanish, but it can become arbitrarily small (see Figure VI).

Note that although F_e' vanishes at $k = 0$, the ratio $F_e / |F_e'|$ is well behaved and vanishes itself at $k = 0$.

Thus, we may write (166) as

$$W_1(0, 1; \omega_L) \cong \int_0^{k_D} dk \, \frac{k_s \, F_e(k = k_s)}{F_e'(k = k_s) + \frac{1}{2}(k - k_s)^2 \left(\frac{\partial^2 F_e'}{\partial k^2} \right)_{k = k_s}} \tag{167}$$

in the neighborhood of this critical wavenumber, as this yields the dominant contribution to $W_1(0, 1; \omega_L)$. Integrating, we have

$$W_1(0, 1; \omega_L) \cong \frac{\pi \sqrt{2} \, k_s \, F_e(k = k_s)}{\sqrt{\Delta} \left| \frac{\partial^2 F_e'}{\partial k^2} \right|^{\frac{1}{2}}} \quad , \tag{168}$$

where for convenience we have written $\Delta = |F_e'(k = k_s)|$. We can easily see that if Δ is sufficiently small, i.e., for an almost unstable distribution, $W_1(0, 1; \omega_L)$ may become extremely large. Collecting our results (156), (161) and (168), we can sketch the behavior of W_1 for the distribution (138) in Figure VII.

The question now arises as to whether there is sufficient area under the spike generated by $\Delta^{-\frac{1}{2}}$ in Figure VII. That is, although the differential intensity $\frac{d^2 I_1}{d\sigma\, d\omega}$ appears to vary with ω in the desired way, there may not be sufficient energy in the lower part of the line for the splitting to be observable. We must consider the fact that an actual detector would, in effect, integrate this differential intensity across a finite frequency band $\Delta\omega$. Thus, we wish to examine whether or not there is sufficient enhancement over such a finite interval.

The motivation for such a query arises from Section 5 of Tidman and Dupree's paper.[5] There the case of an electron beam traversing a Maxwellian plasma was studied. It was found that if only one wavenumber first became unstable (similar in spirit to our case) the additional integrated contribution to the emission was uninteresting (i.e., no large increase). This was true even though the spectral density actually diverged at the unstable wavenumber for this problem. Tidman and Dupree concluded that one must have a range of wavenumbers verging on instability to obtain enhanced emission and we will consider a situation of this type in the next section.

First, however, we wish to study our question about the observed intensity in more detail.

To do this we first note that for small but non-zero θ, the right hand side of (168) becomes a function of θ^2 only. This can easily be seen by referring to (139) and (140) and the definitions of $k_\parallel$ and $k_\perp$. Thus we expand Δ around $\theta = 0$ by writing

$$\Delta(\theta^2) = \Delta(\theta = 0) + \theta^2 \left(\frac{\partial \Delta}{\partial(\theta^2)} \right)_{\theta = 0} \equiv \Delta + \theta^2 A \tag{169}$$

By virtue of the relationship $\omega^2 = \omega_e^2 + \Omega^2 \sin^2 \theta$ (for the fundamental), we may replace our frequency integral $\int d\omega$ by an integral over θ, namely $\int \frac{\Omega^2}{\omega_e} \theta \, d\theta$, which is valid for small θ.

Examination of (168) and (131) for small θ shows that the dominant contribution will indeed come from the $\Delta^{-\frac{1}{2}}$ term in (168) and we need consider only this term. Hence, let

$$J = \int_0^{\theta_c} \frac{\theta \, d\theta}{\sqrt{\Delta(\theta^2)}} = \int_0^{\theta_c} \frac{\theta \, d\theta}{\sqrt{\Delta(\theta=0) + \theta^2 A}} \tag{170}$$

We can see directly from (170) that if $\Delta = 0$, $J = \frac{\theta_c}{\sqrt{A}}$. The exact integral is, of course,

$$J = \frac{1}{\sqrt{A}} \left\{ (\theta_c^2 + \frac{\Delta}{A})^{\frac{1}{2}} - (\frac{\Delta}{A})^{\frac{1}{2}} \right\} , \tag{171}$$

where θ_c represents the upper bound of the spike. To reassure ourselves that the contribution of the spike is not significant, we must check the magnitude of A. As long as A (or $\sqrt{A}$) is not a very small quantity, there will be no spectacular increase of emission.

For convenience, let $\theta^2 = x$, then

$$A = \left[\left| \frac{\partial F_e'}{\partial x} (x, \; k = k_s) \right| \right]_{x=0} \tag{172}$$

and we have for small θ

$$F_e'(x, k_s) \cong -\frac{\beta\omega}{(2\pi)^{\frac{1}{2}} k_s V_e^3} \exp\left[-\frac{\omega^2}{2k_s^2 V_e^2}\right]$$

$$-\frac{(1-\beta)\left\{\omega - k_s U(1-\frac{x}{2})\right\}}{(2\pi)^{\frac{1}{2}} k_s \left\{V_\perp^2 x + V_\parallel^2 (1-\frac{x}{2})^2\right\}^{\frac{3}{2}}} \exp\left[-\frac{\left\{\omega - k_s U(1-\frac{x}{2})\right\}^2}{2k_s^2 \left\{V_\perp^2 x + V_\parallel^2 (1-\frac{x}{2})^2\right\}}\right]$$

$$= F_T'(k_s) + F_E'(x, k_s) \quad . \tag{173}$$

Noting that $F_e'(k_s, x = 0) = \Delta$, we can write

$$A = \left| \frac{(\Delta - F_T')}{(\omega - k_s U)} \left\{ \frac{k_s U}{2} - \frac{3(\omega - k_s U)(V_\perp^2 - V_\parallel^2)}{2V_\parallel^2} - \frac{(\omega - k_s U)^2 k_s U}{2k_s^2 V_\parallel^2} + \frac{(\omega - k_s U)^3}{2k_s^2 V_\parallel^2} \frac{(V_\perp^2 - V_\parallel^2)}{V_\parallel^2} \right\} \right| \tag{174}$$

Since $V_\perp^2 >> V_\parallel^2$, we have

$$A \cong \left| \frac{(\Delta - F_T')}{(\omega_e - k_s U)} \left\{ \frac{k_s U}{2} - \frac{3(\omega_e - k_s U) V_\perp^2}{2V_\parallel^2} - \frac{(\omega_e - k_s U)^2 k_s U}{2k_s^2 V_\parallel^2} + \frac{(\omega_e - k_s U)^3}{2k_s^2 V_\parallel^2} \frac{V_\perp^2}{V_\parallel^2} \right\} \right| \quad , \tag{175}$$

with

$$\left| \frac{\Delta - F_T'}{\omega_e - k_s U} \right| = \frac{(1-\beta)}{(2\pi)^{\frac{1}{2}} k_s V_\parallel^3} \exp\left[-\frac{(\omega_e - k_s U)^2}{2k_s^2 V_\parallel^2}\right] \quad . \tag{176}$$

To estimate the magnitude of A, we need the magnitude of $k_s U$. Our program is to use the definition of k_s as the wavenumber at the minimum of F'_e to obtain its value, and then let $\Delta \rightarrow 0$ to get an upper bound on the U required to just maintain stability. From $\frac{\partial F'_e}{\partial k} = 0$ at $k = k_s$, we obtain

$$k_s^2 \sim k_D^2 (1 - \beta)^{-1} \exp\left(-\frac{k_D^2}{2k_s^2}\right) \quad . \tag{177}$$

For example, if $(1 - \beta) \sim 0(10^{-2})$ this yields $k_s \sim .26\, k_D$. With the requirements that we imposed to find (177), namely that $k_s U \sim \omega_e$ and $V_{\parallel} \sim V_e$, this places $U > \sim 3.8\, V_e$. Now we must verify that this is consistent with our stability requirement that Δ not vanish. The limiting case of $\Delta \sim 0$ with $k_s \sim .26\, k_D$ leads to $(k_s U_{max} - \omega_e)/\omega_e \sim .06$ which certainly satisfies the condition that $k_s U \sim \omega_e$. Thus, $U_{max} \sim 4 V_e$ and we have found a consistent scheme of approximating these transcendental equations which maintains stability.

Returning to (175) and (176), we see that the dominant terms in A give

$$A \sim \frac{(1 - \beta)\, U}{2(2\pi)^{\frac{1}{2}} V_e^3} \left\{ 1 + 3\left(1 - \frac{\omega_e}{k_s U}\right) \frac{V_\perp^2}{V_e^2} \right\} \quad , \tag{178}$$

which is not a small quantity. With our previous numerical values, for example, $A \sim 1.4 - 1.5$.

To review, we have shown that (as in Ref. 5) distributions which have only <u>one</u> wavenumber first verging on instability do not yield sufficient enhancement. The increase of emission over that due to thermal fluctuations is not large enough to give an observable splitting.

F. Results for a Flatter Distribution

Now we turn to a distribution which we hope will provide more enhancement. As discussed previously, we require a whole range of wavenumbers to first verge on instability. Thus we consider a distribution which is flatter than a Maxwellian in $v_\parallel$. We try, therefore, a drifting pancake-like distribution

$$f_e(\underline{v}) = \frac{\beta}{(2\pi)^{\frac{3}{2}} V_e^3} \exp\left(-\frac{v^2}{2V_e^2}\right) + \frac{(1-\beta)}{2\pi V_\perp^2} \exp\left(-\frac{v_\perp^2}{2V_\perp^2}\right) \cdot \frac{I(V_\parallel - |v_\parallel - U|)}{2V_\parallel} \quad . \tag{179}$$

Here, as in section E, $V_\perp^2 >> V_e^2$, $V_\perp^2 >> V_\parallel^2$, and $I(x) = 0$ if $x < 0$, $I(x) = 1$ if $x > 0$. As before, $(1-\beta) << \beta \cong 1$ and all directions are measured with respect to $\underline{B}_0$.

From (179) we obtain

$$F_e(\underline{k}, \frac{\omega}{k}) = \frac{\beta}{\sqrt{2\pi}\, V_e} \exp\left[-\frac{\omega^2}{2k^2V_e^2}\right] + \frac{(1-\beta)k}{4k_\parallel V_\parallel}\left\{\Phi\left(\frac{k_\parallel U + k_\parallel V_\parallel - \omega}{\sqrt{2}\, k_\perp V_\perp}\right) - \Phi\left(\frac{k_\parallel U - k_\parallel V_\parallel - \omega}{\sqrt{2}\, k_\perp V_\perp}\right)\right\} = F_T + F_E \ , \tag{180}$$

where Φ is the error function,

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt\, e^{-t^2} \qquad (181$$

In order to discuss F_E near $\theta = \frac{\pi}{2}$, it is convenient to let $x = \cos\theta$ and make a small x expansion. Designating

$$g(x) = \Phi\left(\frac{k(U + V_{\parallel})x - \omega}{\sqrt{2}\, kV_{\perp}(1 - x^2)^{\frac{1}{2}}}\right) - \Phi\left(\frac{k(U - V_{\parallel})x - \omega}{\sqrt{2}\, kV_{\perp}(1 - x^2)^{\frac{1}{2}}}\right) \tag{182}$$

we have

$$g(x) = g(0) + x\left(\frac{\partial g}{\partial x}\right)_{x=0} + \ . \ . \ . \quad , \tag{183}$$

hence

$$g(x) = \frac{4xV_{\parallel}}{\sqrt{2\pi}\, V_{\perp}} \exp\left(- \frac{\omega^2}{2k^2V_{\perp}^2}\right) \ . \tag{184}$$

Thus, we find

$$F_e(\underline{k}, \frac{\omega}{k})_{\theta \sim \frac{\pi}{2}} = F_T + \frac{(1 - \beta)}{\sqrt{2\pi}\, V_{\perp}} \exp\left(- \frac{\omega^2}{2k^2V_{\perp}^2}\right) , \tag{185}$$

which is identical to (141)-(143). We obtain, therefore, the same result as in section E, namely

$$W_1(0, 1, \omega_U) \cong \frac{2}{3} k_1^3 V_{\perp}^2 \tag{186}$$

where k_1 is defined by (149).

In order to discuss $W_1\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \omega_M\right)$, we can obtain an expression for $F_{E_{\theta \cong \frac{\pi}{4}}}$ based on the point that $k_{\parallel}V_{\parallel} / k_{\perp}V_{\perp}$ is a small quantity. We rewrite F_E so that

$$F_E = \frac{(1 - \beta)}{2\sqrt{2}\, V_{\parallel}} \{ \Phi(y + \Delta) - \Phi(y - \Delta)\} \tag{187}$$

where $y = \frac{kU/\sqrt{2} - \omega}{kV_{\perp}}$ and $\Delta = \frac{V_{\parallel}}{\sqrt{2}\, V_{\perp}} << 1$.

An equivalent expression for (187) is

$$F_E = \frac{1-\beta}{\sqrt{2\pi}\, V_{\parallel}} \left\{ \int_0^{y+\Delta} dt\, e^{-t^2} - \int_0^{y-\Delta} dt\, e^{-t^2} \right\} \tag{188}$$

where we shall let

$$g(y \pm \Delta) = \int_0^{y\pm\Delta} dt\, e^{-t^2} \tag{189}$$

For small Δ, we expand to get

$$g(y \pm \Delta) \cong g(y) \pm \Delta \left(\frac{\partial g\,(y \pm \Delta)}{\partial\,(y \pm \Delta)} \right)_{\Delta=0} \tag{190}$$

Thus, in (188),

$$g(y+\Delta) - g(y-\Delta) \cong \Delta \left\{ \frac{\partial g\,(y+\Delta)}{\partial\,(y+\Delta)} + \frac{\partial g\,(y-\Delta)}{\partial\,(y-\Delta)} \right\}_{\Delta=0}, \tag{191}$$

and we find

$$F_E \cong \frac{(1-\beta)}{\sqrt{\pi}\, V_{\perp}} \exp\left[- \frac{(kU/\sqrt{2} - \omega)^2}{k^2 V_{\perp}^2} \right] \tag{192}$$

We see that (192) is exactly the F_E which is obtained from (139) when $\theta = \frac{\pi}{4}$ and $V_{\perp}^2 >> V_{\parallel}^2$. Thus we simply quote the results of section E to the effect that

$$W_1\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}; \omega_M \right) \cong \frac{1}{2} W_1(1, 0; \omega_U) \tag{193}$$

We are left with the task of examining the situation for $W_1(0, 1; \omega_L)$ (i.e., $\theta \sim 0$). From either direct examination of (180) for $k_{\perp} \to 0$, or from (111) with $\delta(u - v_{\parallel})$, we obtain

$$F_E = \frac{(1-\beta)}{2V_\parallel} I\left(V_\parallel - \left|\frac{\omega}{k} - U\right|\right) \quad , \tag{194}$$

where I is the step function used in (179). For $k_\perp = 0$, we have

$$F_E' = -\frac{(1-\beta)}{2V_\parallel}\left\{\delta\left(\frac{\omega}{k} - [V_\parallel + U]\right) - \delta\left(\frac{\omega}{k} + [V_\parallel - U]\right)\right\}. \tag{195}$$

In the region $k < k_1$ where

$$k_1 \cong \frac{k_D}{\sqrt{2}}\left\{\log\left[\sqrt{2/\pi}\ \frac{V_\parallel}{V_e(1-\beta)}\right]\right\}^{-\frac{1}{2}} \quad , \tag{196}$$

we can approximate $F_e / |F_e'|$ by

$$\frac{F_e(\underline{k}, \frac{\omega}{k})}{|F_e'(\underline{k}, \frac{\omega}{k})|} \cong \frac{(1-\beta)\, k V_e^3 \sqrt{\pi}}{\sqrt{2}\ V_\parallel\ \omega}\ I\left(V_\parallel - \left|\frac{\omega}{k} - U\right|\right) \exp\left(\frac{\omega^2}{2k^2 V_e^2}\right) . \tag{197}$$

Similarly,

$$\frac{F_e(-\underline{k}, \frac{\omega}{k})}{|F_e'(-\underline{k}, \frac{\omega}{k})|} \cong \frac{(1-\beta)\, k V_e^3 \sqrt{\pi}}{\sqrt{2}\ V_\parallel\ \omega}\ I\left(V_\parallel - \left|\frac{\omega}{k} + U\right|\right) \exp\ \frac{\omega^2}{2k^2 V_e^2} \quad . \tag{198}$$

Note that the superthermal particles do not contribute to the damping in this case.

Using (197) and (198) in (132) we have

$$W_1(0, 1; \omega_L) \cong \frac{(1-\beta) V_e^3 \sqrt{\pi}}{\sqrt{2}\ V_\parallel\ \omega}\left\{\int_{k_2}^{k<k_1} k^2\, dk \exp\left(\frac{\omega^2}{2k^2 V_e^2}\right) + \int_{k_3}^{k<k_1} k^2\, dk \exp\left(\frac{\omega^2}{2k^2 V_e^2}\right)\right\} \quad , \tag{199}$$

where $k_2 = \dfrac{\omega}{V_{\parallel} + U}$ and $k_3 = \dfrac{\omega}{V_{\parallel} - U}$. From our requirements, these are very narrow regions of integration. The dominant factor is, of course, the exponential; k does not change very much throughout the region by comparison. We do not incur too large an error if we write

$$W_1(0, 1, \omega_L) \cong \frac{1 - \beta V_e^5 \sqrt{\pi}}{\sqrt{2}\, V_{\parallel}\, \omega^3} \left\{ k_2^5 \int_{x_1}^{x_2} dx + k_3^5 \int_{x_1}^{x_3} dx \right\} , \tag{200}$$

where $x = \exp\left(\dfrac{\omega^2}{2k^2 V_e^2}\right)$ and x_1, x_2, and x_3 have the respective k's in the argument of the exponential. Since $x_2 >> x_3 >> x_1$, the dominant term is clearly

$$W_1(0, 1, \omega_L) \sim \frac{(1 - \beta)\, k_2^5 V_e^5 \sqrt{\pi}}{\sqrt{2}\, V_{\parallel}\, \omega^3} \exp\left(\frac{\omega^2}{2k_2^2 V_e^2}\right) . \tag{201}$$

Thus, from (131) and the exponential in (201) we see that $\dfrac{d^2 I_1}{d\sigma\, d\omega}$ can be greatly enhanced at the lower edge of the line. Note that, as in section E, all our preceding and subsequent general arguments apply equally well to the second harmonic.

Again, as in the case for the drifting Maxwellian, we must confirm whether or not there is sufficient area under the W_1 curve (i.e., the total energy) for the splitting to be observable.

We are interested, therefore, in the effect of taking θ to be small but finite and integrating across a finite $\omega \leftrightarrow \theta$ interval. We make a small θ expansion of the θ-dependent function in (180). Designate $h(\theta)$ such that

$$h(\theta) = \left(1 - \frac{\theta^2}{2}\right)^{-1} \left\{ \Phi\left(\frac{(U + V_\parallel)(1 - \frac{\theta^2}{2}) - \frac{\omega}{k}}{\sqrt{2}\, V_\perp\, \theta}\right) - \Phi\left(\frac{(U - V_\parallel)(1 - \frac{\theta^2}{2}) - \frac{\omega}{k}}{\sqrt{2}\, V_\perp\, \theta}\right)\right\} \tag{202}$$

We use

$$h(\theta) \cong h(0) + \theta\left(\frac{\partial h}{\partial \theta}\right)_{\theta=0} + \theta^2\left(\frac{\partial^2 h}{\partial \theta^2}\right)_{\theta=0} , \tag{203}$$

since it turns out that $(\partial h/\partial\theta)_{\theta=0}$ is zero. Thus,

$$h(\theta) \cong I(V_\parallel - |\tfrac{\omega}{k} - U|) + \theta^2 \Big\{ I(V_\parallel - |\tfrac{\omega}{k} - U|) - (U + V_\parallel)\delta(U + V_\parallel - \tfrac{\omega}{k}) + (V_\parallel - U)\delta(V_\parallel - U + \tfrac{\omega}{k})\Big\} . \tag{204}$$

Since (197) now becomes

$$\frac{F_e(\underline{k}, \frac{\omega}{k})}{|F'_e(\underline{k}, \frac{\omega}{k})|} \cong \frac{(1-\beta)\, k V_e^3 \sqrt{\pi}}{\sqrt{2}\, V_\parallel\, \omega} \exp\left(\frac{\omega^2}{2k^2 V_e^2}\right) h(\theta) , \tag{205}$$

substitution of (205) in (132) yields

$$W_1(\theta, 1; \omega_L) \cong W_1(0, 1; \omega_L) \left\{ 1 + \theta^2 \left[1 - \left(\frac{U + V_\parallel}{V_e}\right)^2 \right] \right\} . \tag{206}$$

Assuming $(U + V_\parallel)^2 >> V_e^2$, we have

$$W_1(\theta, 1; \omega_L) \sim W_1(0, 1; \omega_L) \left[1 - \theta^2 \left(\frac{U + V_\parallel}{V_e}\right)^2 \right] . \tag{207}$$

As in section E, we now integrate across a finite frequency interval with $\int d\omega \propto \int \theta\, d\theta$. Define K such that

$$K = \int_0^{\theta_0} \theta\, d\theta\, W_1(\theta, 1; \omega_L) , \tag{208}$$

where θ_0 measures the extent of the spike in W_1. Hence,

$$K \sim W_1(0,1;\ \omega_L)\,\frac{\theta_0^2}{2}\left\{1 - \frac{\theta_0^2}{2}\left(\frac{U + V_{\parallel}}{V_e}\right)^2\right\} . \tag{209}$$

For $(U + V_{\parallel})^2/V_e^2$ of the order of 25, for example, we see from (209) that K is still greater than one half of the value obtained by assuming <u>no</u> decrease with θ (i.e., $K_0 = W_1(0,1;\ \omega_L)\theta_0^2/2$), even at $\theta_0 \sim 10^0$.

We have confirmed, therefore, that this case generates a large increase (compared to thermal) in the total energy emitted in the lower frequency part of the line (see Figure VII). The splitting thereby becomes clearly observable.

To recapitulate, the essential difference between distributions (138) and (179) is that (138) leads to a situation in which only one wavenumber $k = k_s$ can first verge on instability, whereas (179) allows a whole range of $\underline{k}$'s to first border on instability. The flatness of f_E in (179) indicates that the superthermal electrons in section F do not contribute to the Landau damping of the plasma oscillations whereas the energetic electrons considered in section E do. Hence, the greatly increased emission for the section F case.

G. Application to Type II Solar Radio Bursts

Type II solar radio bursts have two main characteristics, as exhibited in a frequency versus time plot (see Figure VIII). One feature is the presence of two broad frequency bands, with the upper band at approximately twice the frequency of the lower band. The other major feature to be noted is the decrease in frequency of each band over a period of several minutes.

The frequency of the lower band has been interpreted as being associated in some way with the plasma frequency $(\omega_e^2 = 4\pi e^2 n_0/m)$ and that of the upper band with $2\omega_e$, hence their designation as the fundamental and second harmonic respectively. The diminution in frequency is then taken to reflect a lower density (n_0) in the source region with increasing time. The picture which emerges, therefore, is the commonly accepted one of some disturbance propagating up through the solar corona into regions of lower and lower density and hence, lower and lower ω_e. If one picks a model for the variation of coronal density versus altitude above the photosphere (e.g., the Baumbach-Allen[24,25] values) an estimate can be made of the velocity of this disturbance. This value turns out to be of the order of 1000-1500 Km/sec, i.e., slightly sub-thermal with respect to the electron thermal velocity V_e, but super-sonic with respect to ion thermal speeds (note $T_{corona} \cong 10^6$ °K). For this reason, one expects that Type II's originate from a plasma wave of some sort, perhaps a shock wave. These Type II events, then, are often denoted as "slow-drift" bursts, in contradistinction to Type III's which are apparently characterized by bursts of relativistic electrons traveling in a stream.

Specifically, we now consider the model discussed recently by Tidman[26] in which it is assumed that a collisionless shock wave is generated by the upward expansion of plasma from a flare site. In the absence of a guiding theory of plasma shock structure, it is assumed that the coronal shock wave has some of the features experimentally observed in the Earth's bow shock wave in the solar wind (Ness, Scearce, and Seek[27]). The shock transition, therefore, is likely to be turbulent in nature and to propagate at some arbitrary angle to any relatively ordered magnetic field which exists ahead of the front.

In the turbulent region behind the collisionless bow shock front of the Earth, fluxes of energetic electrons have been observed. The generating mechanism for such fluxes is not yet clear, although some kind of stochastic acceleration process in the region of turbulence behind the front may be involved. We assume that in the collisionless coronal shock wave, a similar condition of disordered plasma and magnetic field together with a flux of energetic electrons also exists (see Figure IX). We shall now consider our previous calculations in the context of these assumptions to see how many of the features of Type II events we can explain.

It has been shown by Tidman[26] that with plausible fluxes of superthermal electrons in the excited plasma, total intensities for the two-harmonic plasma radiation can be obtained which are in agreement with those measured in a Type II disturbance. This calculation assumed isotropic electron velocity distributions $f_e(|\underline{v}|)$ and no magnetic field.

If we now assume that the superthermal electron distributions are sometimes of the "drifting-pancake" type as in (179), then we have shown that substantial amounts of radiation are indeed generated at $\sim \omega_e$ and $\sim 2\omega_e$ with a splitting that is about $\Omega^2/2\omega_e$ for the fundamental and Ω^2/ω_e for the second harmonic. From the observed splitting[25] in a typical Type II event, magnetic field intensities can be calculated for the source region (see Table I). The values seem of the right order of magnitude although they may be a little high (for the validity of the weak field approximation used in our calculations), particularly for the fundamental. The altitudes above the photosphere were calculated assuming that the source region is propagating out along a coronal streamer (following a field line) in which the local electron density is 10 times the Baumbach-Allen values.[24,25]

TABLE I

Magnetic Field in Source Region Deduced from Line Splitting of the Fundamental

Time	Observed $\frac{\omega_e}{2\pi}$, Mcs	Observed $\frac{\Delta\omega}{2\pi}$, Mcs	Electron Density n_0, cm^{-3}	Magnetic Field, Gauss	Height above Photosphere, Km
↓	120	25	1.8×10^8	27	3×10^5
	80	10	8×10^7	14	4×10^5
	40	5	2×10^7	7	7×10^5

We note that Type II's occur much less frequently than Type III's and it seems reasonable to expect that anisotropic distributions of the kind proposed in (179) sometimes exist in the source region behind the shock front. Cyclotron waves propagating up the ordered field lines behind the shock wave, when they first penetrate into the turbulent region, accelerate the electrons in a plane perpendicular to $\underline{B}_0$ and thereby feed energy into the $\underline{v}_\perp$ component of the electron velocities. Thus the superthermal electrons could be produced with $V_\perp^2 \gg V_\parallel^2$. Electromagnetic radiation, especially at $\omega \sim \omega_e$, emitted by the excited plasma in the rearward direction will be reflected by regions of higher density. Thus, as it passes back through the source region, part of the energy will be available to accelerate the energetic electrons.

In the tangled magnetic field immediately behind the front some energetic electrons would be deflected into the parallel direction. Those closest to the front will escape through it and up the ordered field lines. This will tend to produce a drift of the superthermal population relative to the thermal electrons behind the front. It is natural to ask why the energetic electron distribution function should only <u>verge</u> on instability, isn't it likely to actually be unstable over a range of wavenumbers for the situation we are considering? It seems to be a question of time scales. If a distribution is initially unstable then in a very short time unstable waves have grown to such an amplitude that they rapidly drive the distribution function back to stability. If, for example, $f_e(v_\parallel)$ were at some time double-humped (i.e., unstable), we would expect that in only a few periods (ω_e^{-1}) the bump on the tail would be flattened out enough so that the distribution is now only verging on instability over a wavenumber range. This type of flat distribution appears to be the natural result of a drifting instability, as suggested by quasilinear theory, and may be a state in which the superthermal distribution spends most of its time.

In applying our calculations to Type II events, the following points should also be noted:

(a) Any small amount of polarization which might be originally present in the generated electromagnetic radiation will be almost completely destroyed as the radiation leaves the turbulent source region;

(b) the probability of three electron plasma oscillations combining (so as to satisfy energy and momentum considerations) to produce electromagnetic radiation at $3\omega_e$ would seem to be extremely low.

Thus, a model of the type discussed here and in the earlier paper by Tidman[26] can satisfactorily explain the following features of Type II solar radio bursts: unpolarized radiation with a two-harmonic structure and no detectable third harmonic, the total intensity, and the line splitting.

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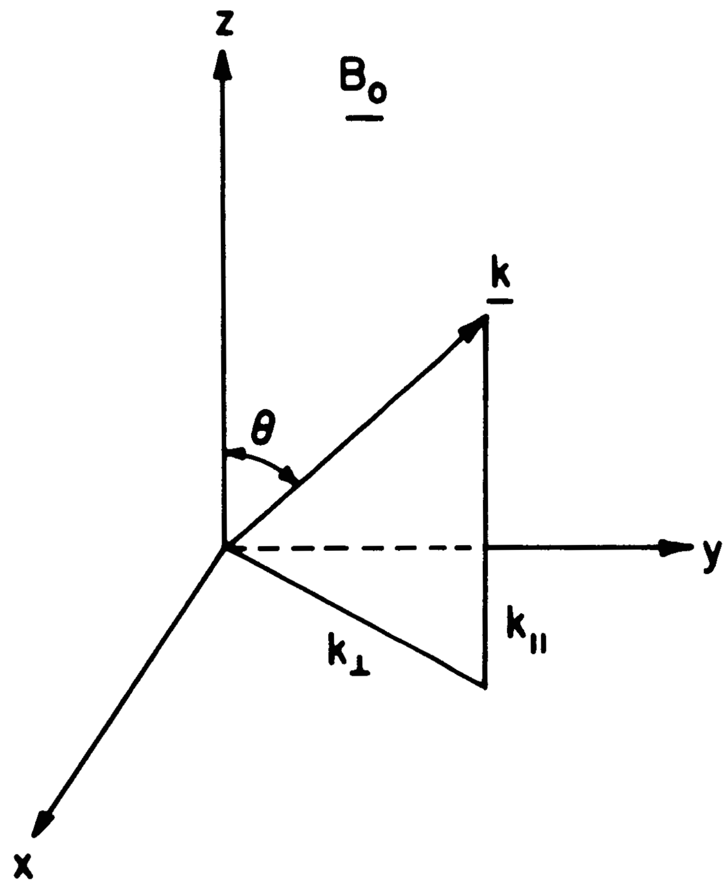


Figure I. Coordinate System with a Magnetic Field.

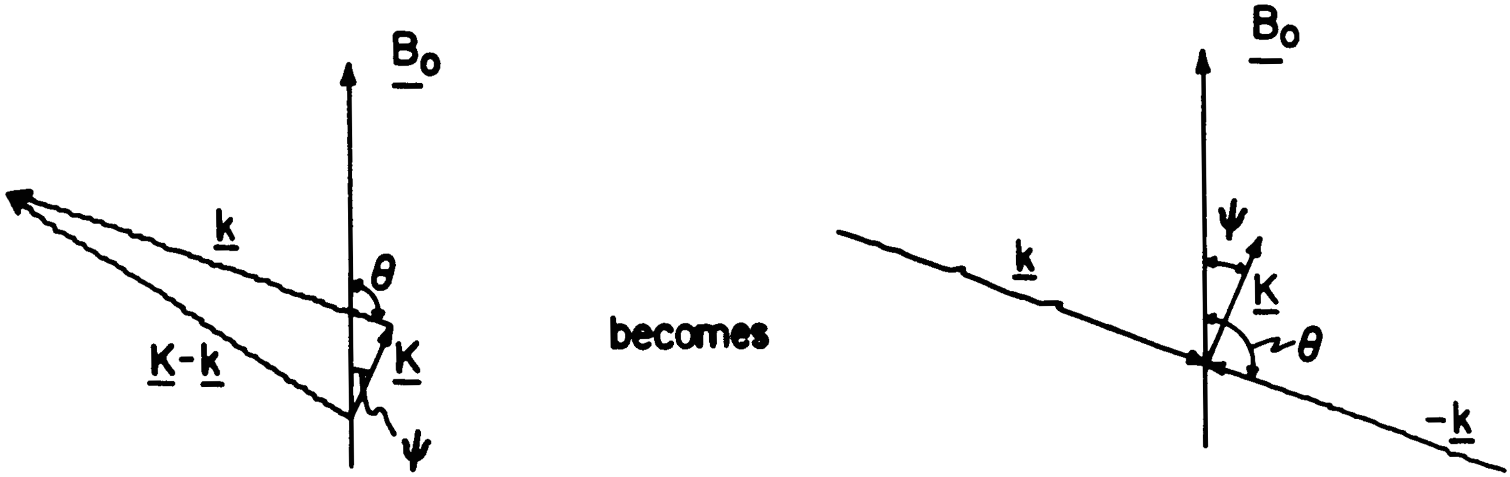


Figure II. Small Electromagnetic Wavenumber Approximation.

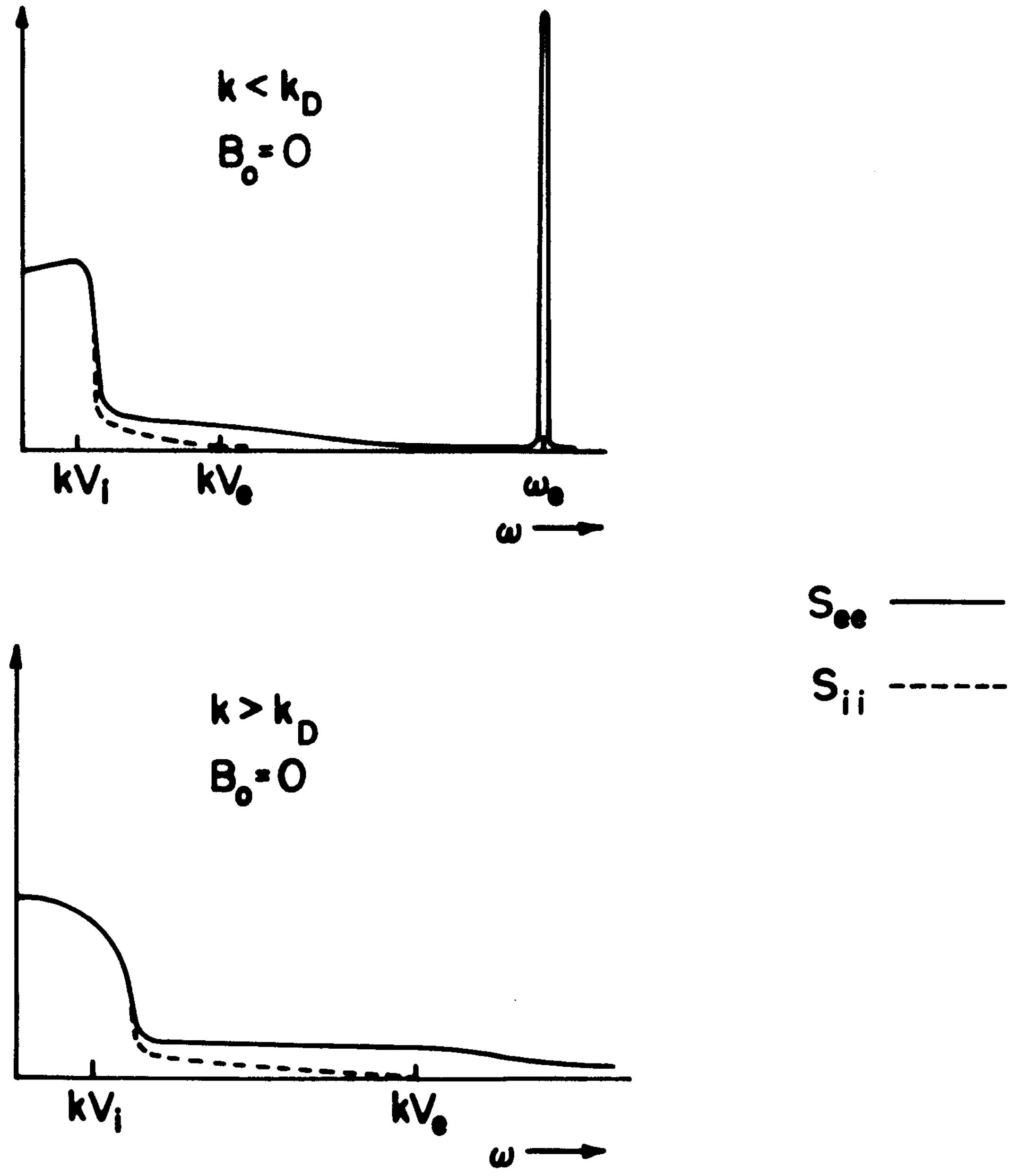


Figure III. Schematic Plot of Electron and Ion Spectral Densities as Functions of Frequency.

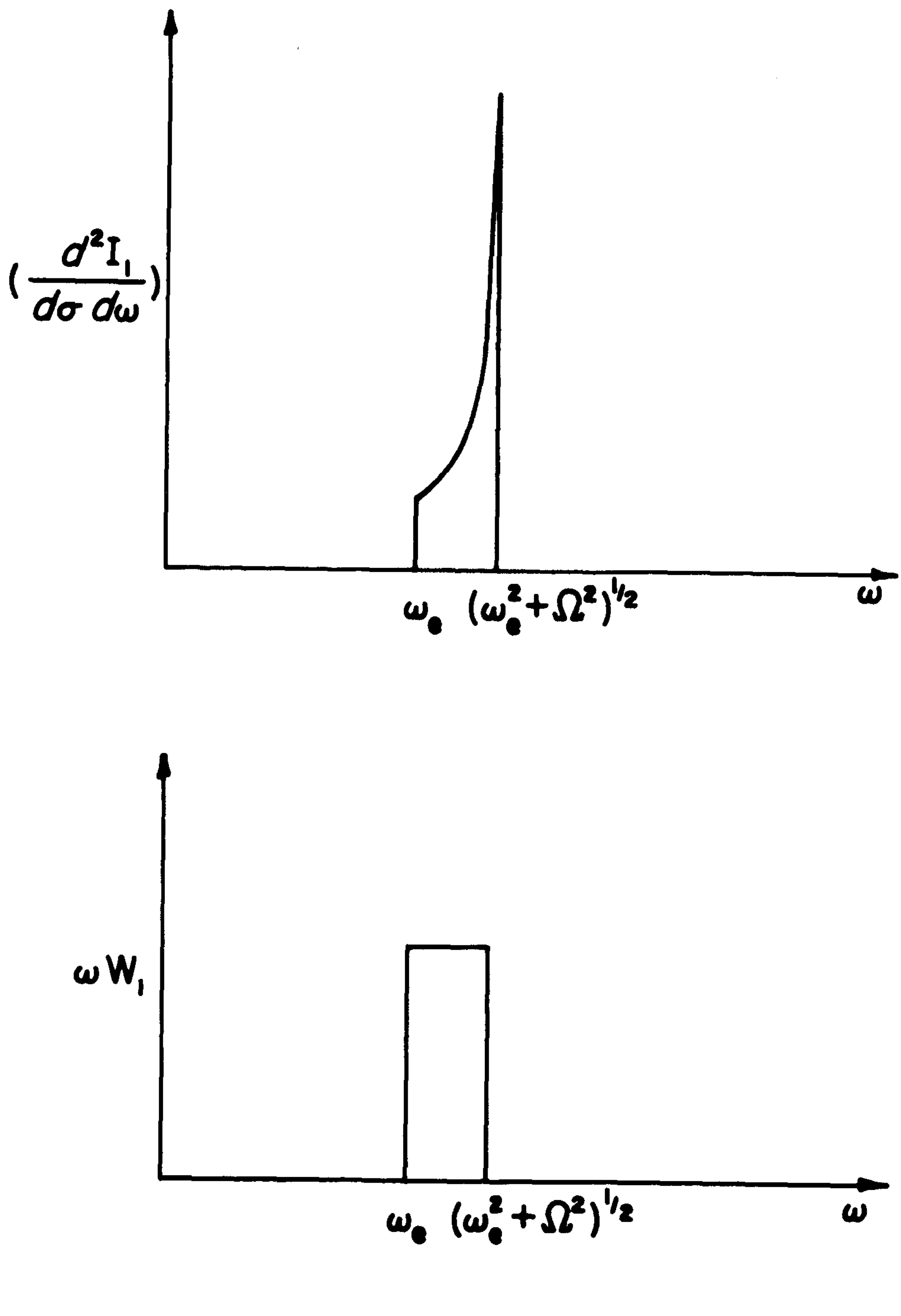


Figure IV. Weighting Function and Differential Intensity for the Fundamental using a Typical Isotropic Distribution.

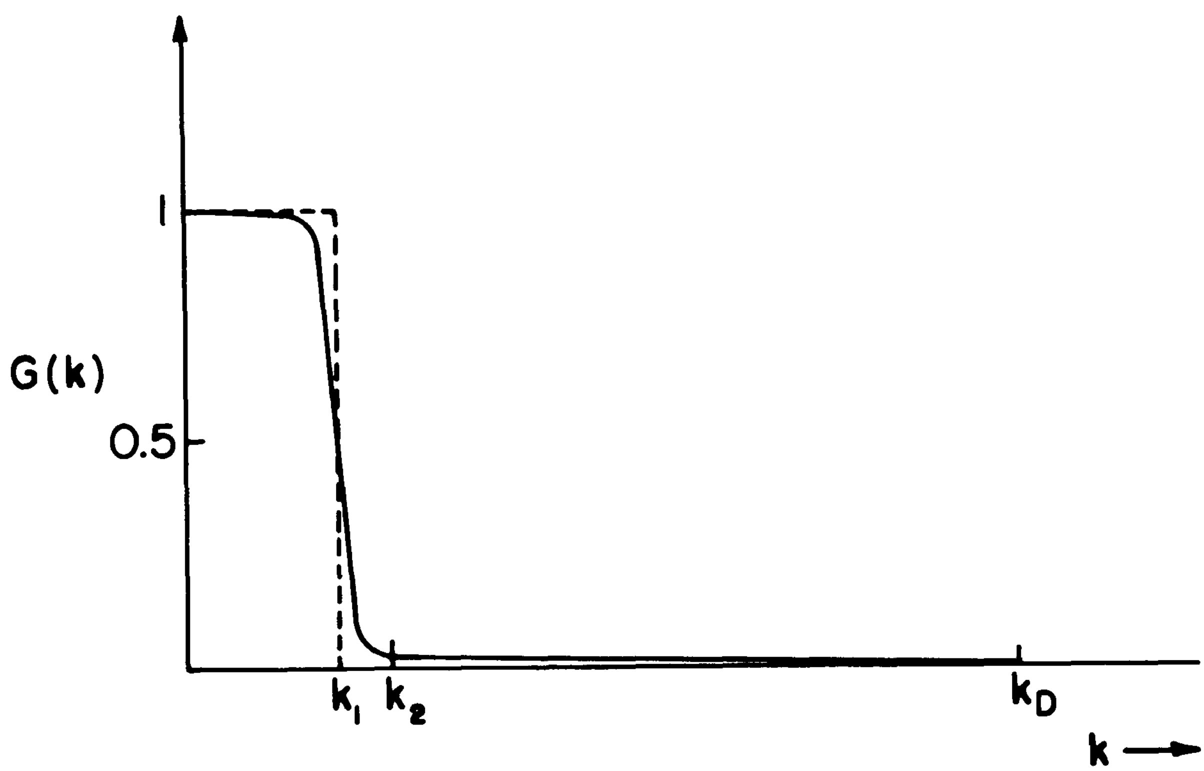


Figure V. A Schematic Representation of a Factor in the Integrand of $W_1(1, 0; \omega_u)$.

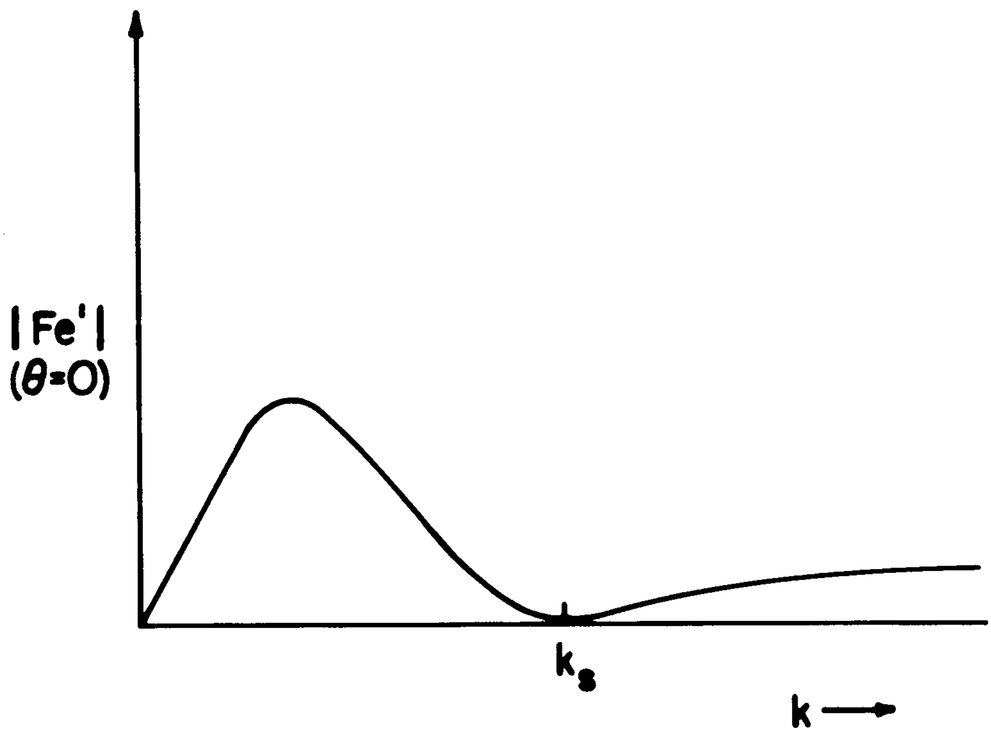
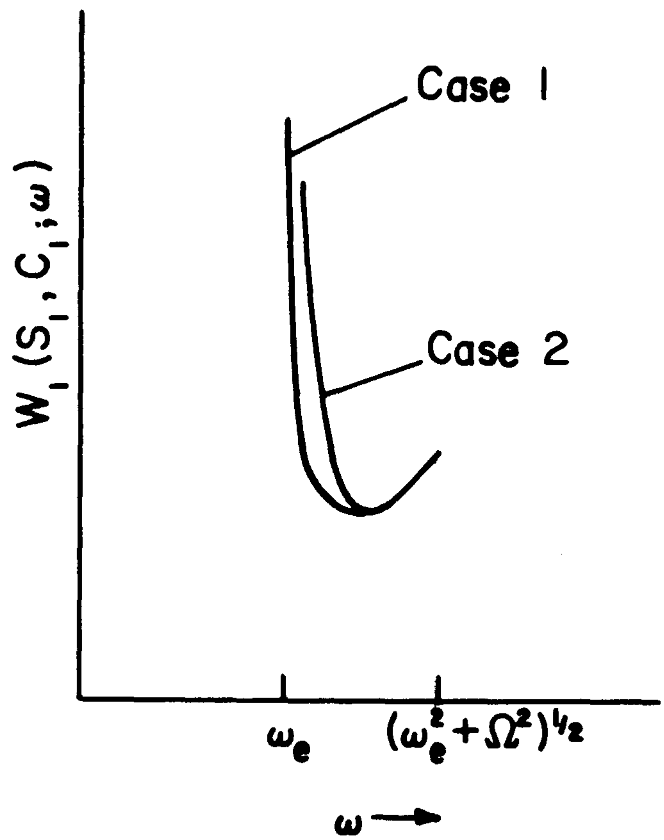


Figure VI. Schematic Variation of $|Fe'|$ at $\theta = 0$ versus k for a Drifting Maxwellian.



Case 1. A Drifting Maxwellian
Case 2. A Flatter Distribution

Figure VII. Weighting Function versus Frequency for Two Anisotropic Distributions.

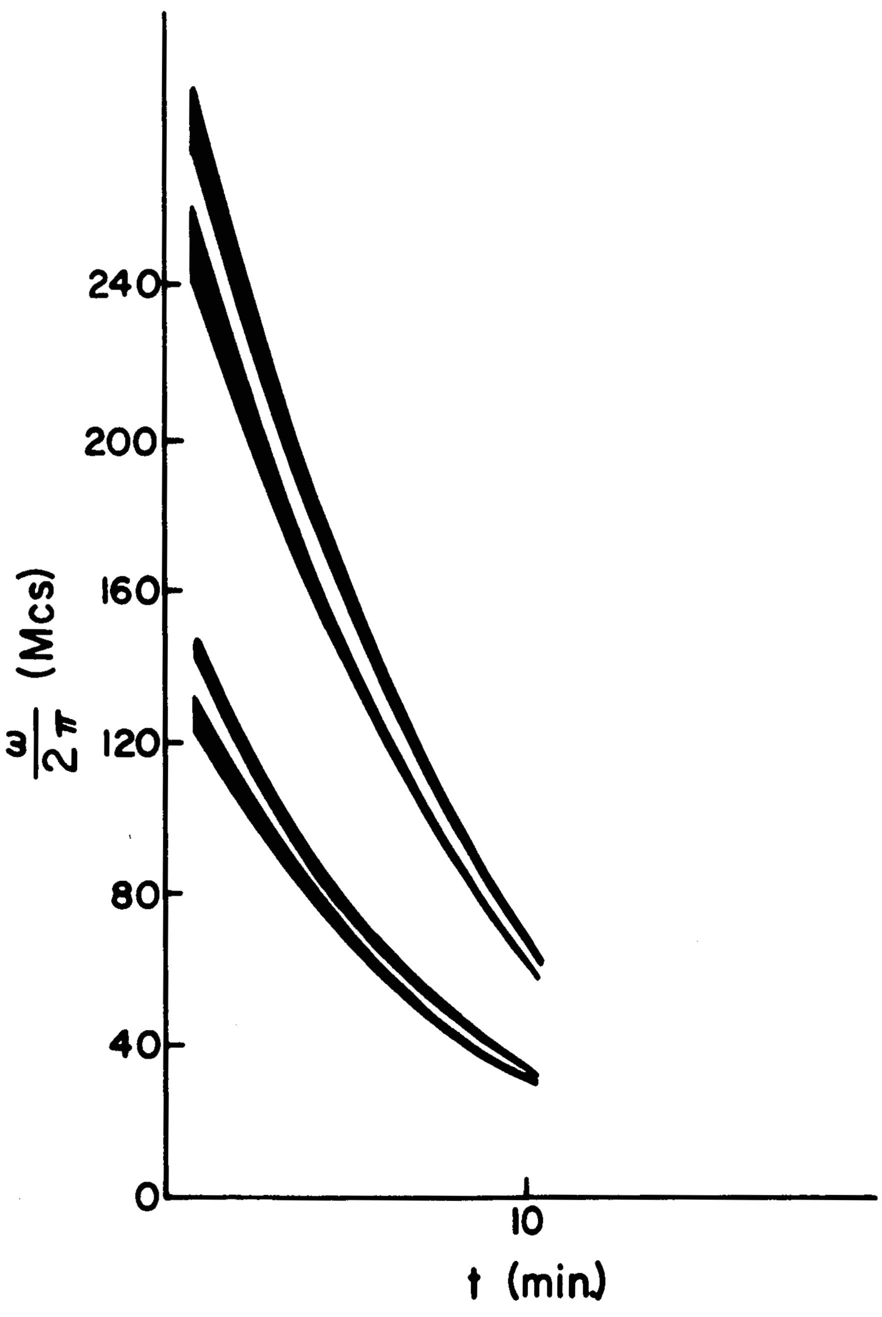


Figure VIII. Idealized Plot of Frequency versus Time for a Type II Solar Radio Burst.

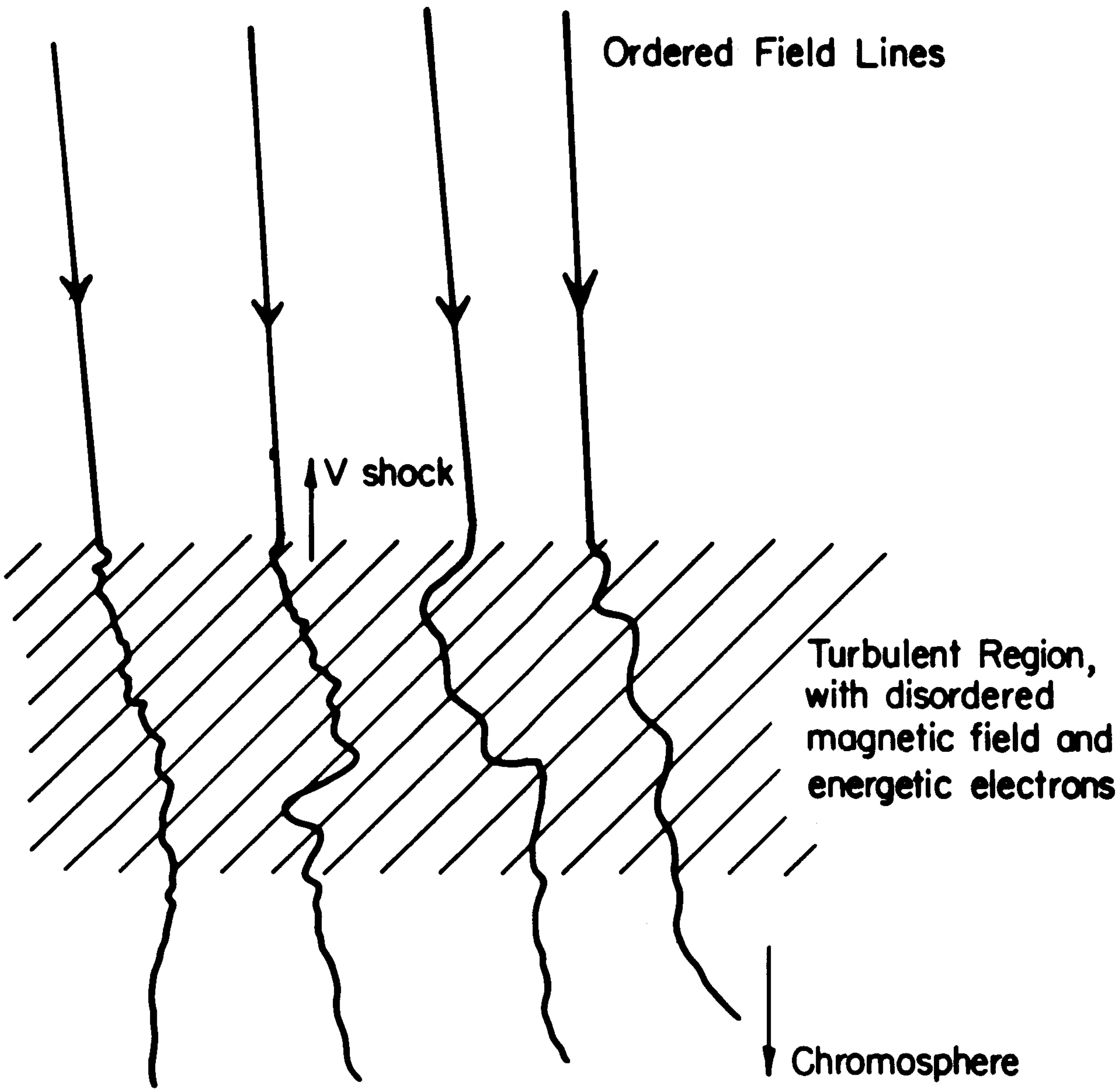


Figure IX. Idealized Sketch of a Collisionless Shock Wave in the Solar Corona.